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On the Contents and Significance of the Khāqānī Zīj by Jamshīd Ghiyāth al-Dīn al-Kāshī

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Preface

This is a description of a Persian zīj, a manual for the practicing medieval astronomer-astrologer. Well over two hundred such sets of tables are known to have existed (most of them written in Arabic), of which about a hundred and twenty are extant. They may well be the most useful single category of sources for the history of the exact sciences during the Middle Ages, and indeed nine zījes have been published. Nevertheless, there are special reasons for publishing also the Khāqānī Zīj.

Its author, Jamshīd al-Kāshī, was the first director of the observatory which made Samarqand, for a few years, the scientific capital of the world. His contributions to computational mathematics marked the culmination of a steady advance in the field which was characteristic of mathematics in the world of medieval Islam. Unlike his contemporaries, who continued to calculate using a bastard system employing a mixture of sexagesimal fractions combined with non-place value alphabetical integers (the *abjad* numerals), plus the Indian place value decimal integers, Kāshī computed with pure sexagesimals. His sine table, for instance, is calculated at intervals of a minute of arc, hence it has 5400 (=90×60) entries. Each entry is precise to four sexagesimal places. This is roughly equivalent to eight decimal places, a remarkable if impractical degree of precision. A number of the other tables in the zīj are to four sexagesimal places. Unlike most zīj writers, Kāshī presented proofs for algorisms, so that his work is also a source for the history of trigonometry.

Here also for the representation of numbers, the mixed system will be employed. Instead of a period, customary for the decimal system, a semicolon will be used as a "sexagesimal point" to separate fractions from integers, commas will be used to separate sexagesimal integers.

Contents and Organization

The zīj consists of a preface, followed by six treatises *(maqālāt)*. The preface dedicates the work to the Sultan Ulugh Beg (d.1449), a grandson of Tīmūr (Tamerlane), ruler of the region since called Uzbekistan, founder of the Samarqand observatory, and a scientist in his own right. There is also a report and analysis of three lunar eclipses observed by the author at Kashan, Iran.

The first treatise describes five calendars then in common use. The second presents the standard trigonometric and astronomical functions. The third, by far the largest, describes all the current procedures for calculating positions and motions of the sun, moon, and five planets. The fourth solves problems of spherical astronomy. The last two treatises are directed toward astrological ends, the fifth giving many solutions of a single problem, the determination of the ascendent (horoscope), whereas the sixth describes a number of indicators: the *tasyīr*, the *fardār*, and the *intihā'*, each having several varieties. These indicators were supposed to enable the prediction of events in the lives of individuals and in the world at large.

The treatises tend to have a standard format. They have an introduction and two chapters. The introduction is usually a useful glossary of technical terms used in the treatise. Chapter 1 is divided into numbered sections (*fuṣūl*, sing. *faṣl*) giving rules for solutions and computations, sometimes with examples of numerical results. It is convenient and illuminating to display such expressions here in algebraic symbolism, but of course there is no trace of symbolism in the original text. Rules are written out verbally. Chapter 2 consists of proofs, also usually in numbered sections, sometimes subdivided into numbered rules (*qāᶜidāt*, sing. *qāᶜida*). Unfortunately the same section number in the two chapters does not usually correspond to the same subject. For cross reference purposes below, passages in the text are indicated by combinations of numbers: the treatise in Roman numerals, then the chapter, then the section numbers, preceded by a semicolon. Note that the introductions to the treatises have no numbered sections. References to the tables are given as folio numbers of the India Office manuscript. Numbers in square bracket are references to the bibliography at the end of the paper.

The Author of the Zīj

Jamshīd Ghiyāth al-Dīn al-Kāshī (or al-Kāshānī) was a native of Kashan, but he seems to have spent a good part of his life in other parts of Iran. For a time his patron at Shiraz was another grandson of Tīmūr, Iskandar b. ᶜUmar Shaykh, the ruler of Fars ([38], p.105). Kāshī himself mentions having been in Isfahan, ([10], p.176), but by the time of the eclipses of 1407 he was back in Kashan. He was also there in 1416 when he completed the first version of his *Nuzhat al-Ḥadā'iq*, a description of an equatorium, an analog computer he invented for determining planetary positions. By 1420 he must have joined the team of scientists Ulugh Beg had assembled at Samarqand to build and operate the observatory. In 1424 Kāshī completed *al-Risālat al-Muḥīṭiya* [11] his unprecedentedly precise determination of the number π, a masterpiece of computational technique. Three years later he finished another major work, the *Miftāḥ al-Ḥisāb* [10]. This covers the whole field of arithmetic, and contains the earliest complete description of operations with decimal fractions.

On the morning of Wednesday, 19 Ramaḍān, 832 H (22 June 1429), at the observatory, Kāshī died, leaving incomplete the observations required for Ulugh Beg's zīj. His successor as director, Qāḍīzāda al-Rūmī, also died with the work unfinished. Their much younger colleague, ᶜAlī Qūshchī, took over and succeeded.

Manuscripts of the Zīj

India Office (London) MS 430 (Ethé 2232)

The material presented below is based principally upon a microfilm of this manuscript. References to it give the folio and line numbers, separated by a colon.

Aya Sofya (Istanbul) MS 2692

References to this manuscript are preceded by AS. These two manuscripts seem to represent two versions rather than identical copies. Sentences in one or the other have frequently been rephrased, presumably by the author. There is a slight indication from the simplified tables of latitudes for Mercury in Treatise III that AS may be the earlier version, however this is anything but conclusive.

Other copies are

Hyderabad, Asafiya MS 323 (an excellent copy)
Dar al-Kutub (Cairo), MS T(aymur) R(yada) 149
Jaipur Maharaja's Library, MS 9
Hyderabad, Asafiya MS 305 (defective)
Leiden, MS Or. 14 (a short excerpt only, on correcting the lunar motions)

DESCRIPTION OF THE ZIJ

f.1r Title Page

On the front of the flyleaf preceding folio 1 is a note reporting the date of Kāshī's death as given above.

f.1v:1 (AS 0v:1) **Invocation and Dedication**

The book opens with elaborate thanks and praise to the Creator of the terrestrial sphere and the surrounding heavens, then passing on to the Prophet Muḥammad, his family and companions. The author now names himself as having long labored in the fields of science until so much material accumulated that he felt it incumbent to compose a zīj. This he set about, much of the time secluded in a house in Kashan, and discouraged. But "the sun of graciousness burst shining from the horizon of happiness", and so on for more than a page of extravagant figures of speech, many of them astronomical, all lauding the virtues of the Khāqān Ulugh Beg. Being accepted at his court, Kāshī completed the zīj and presented it to the imperial library.

The act is likened to that of an ant who (as related in two poetic quotations, one Arabic, the other Persian) gave to Solomon a locust's leg, saying "the gift is proportioned to the size of the giver".

f.2v:1 (AS 1v:7) **A Statement as to How the Zīj Was Composed**

The author decided that, except for the lunar parameters, he would adopt the mean motions used in the Zīj-i Īlkhānī, these having been obtained from observations directed by Nasīr al-Dīn al-Tūsī (d.1274) at the Maragha observatory in western Iran. The equations, however, would be derived anew, and various improvements adopted so that, for instance, the true longitudes of Mars would come out sometimes with a difference of almost a degree as between the two zījes.

f.3r:4 Therefore for this work the name (AS 2r:13) *Zīj-i Khāqānī dar Takmīl-i Zīj-i Īlkhānī* (The Khāqānī Zīj in Completion of the Īlkhānī Zīj) would be adopted.

f.3r:6 (AS 2r:18) **A Numbered List** comprising seventy specific topics demonstrating the superiority of this zīj over the Zīj-i Īlkhānī and its commentaries in particular, and to all other zījes in general.

Examples are:

2. Glossaries of technical terms.
3. Presentation of proofs.

9. A table for converting from the Yazdigird to the Malikī calendar.
10. Correction of certain values in the sine table.
40. A new method of predicting solar eclipses.
45. Corrections in the planetary arcs of visibility.
60. Determination of the ascendent from the shadow cast by a peg in a wall.
61. Determination of the ascendent from the observation of two stars which have the same azimuth.
69. Responses to criticisms made by commentators to the Īlkhānī Zīj.

f.4r:15 (AS 3r:22) **Derivation of the Lunar Mean Motions Based Upon Observations of Three Lunar Eclipses from the City of Kashan**

The dates of the eclipses were

30 Shahrīvar (Old Style) 775 Yazdigird = 2 June 1406, (Oppolzer 4043)
27 Isfand 775 Y = 1 November 1406, (Oppolzer 4044)
18 Shahrīvar 776 Y = 21 May 1407, (Oppolzer 4045)

The method of Ptolemy is used, with minor variations. All the calculations are reported in full. Among the results are, for the lunar daily mean motion:

$$13;10,35,1,52,47,50,50^{\circ}.$$

This is very close to the value

$$13;10,35,1,52,46,45^{\circ}$$

given by Muḥī al-Dīn al-Maghribī, who participated in the Maragha observations.
For the daily anomalistic mean motion Kāshī has

$$13;3,53,56,30,37,20^{\circ}.$$

This is very close to

$$13;3,53,56,29,38,38^{\circ},$$

Ptolemy's *uncorrected* value given in the Almagest ([35], vol.2, p.204).

f.6r:19 (AS 5v:17) A list, giving the titles of the six treatises of which the zīj is composed.

f.6v:9 (AS 6r:1) TREATISE I. ON CALENDARS

(Dr. Benno van Dalen, Institut für Geschichte der Naturwissenschaften, Robert-Meyer-Str.1, Frankfurt am Main, D-60325, Germany, has written a DOS computer program named CALH which includes the five calendars described in this treatise, as well as additional medieval ones and the common modern calendars. If a date in any one of these is entered, the program immediately displays the corresponding Julian day number, the day of the week, and the equivalent date in all of the other calendars.)

f.6v:10 **Introduction: On Determining Years, Months, and Days**

Brief definitions of fundamental concepts are given: solar and lunar years, months, and days, and the relations between them. The notion of eras dating from memorable events is introduced.

f.7r:1 (AS 6r:14) **Chapter 1. On Seleucid (Rūmī), Hijrī, and Yazdigird Dates, and the Extraction of One from the Other**

For each of the three calendars mentioned in the title, year lengths are given, the names and lengths of the months, and any peculiarities of the particular calendar. The initial week day (*madkhal*) of the era of each is given, as well as the number of days, in decimals and sexagesimals, between pairs of eras.

f.7v (AS 7r) **Tables of the Elevated Days of the Hijrī, Seleucid, and Yazdigird Calendars**

For years 1, 2, 3, ..., 60, 120, 180, ..., 1860 of the three calendars named, the entries give the days from each epoch (in decimals and pure sexagesimals) and the *madkhal* (initial day of the week) or increments of madkhals.

f.8r **Table of the Elevated Days of the Madkhal Arguments of the Three Calendars**

With initial month days of the same calendars as argument, the entries are as defined for the preceding tables.

f.8v:11 (AS 8r:1) **Chapter 2. On the Chinese-Uighur Calendar**

(This calendar, widely used in the Mongol empire, is explained in [8]. The earlier publication, [22], is incomplete.)

Section 1. On the Determination of Year and Month Beginnings

Given are definitions of the *fên* (a ten thousandth of a day), and other Chinese units, their transcribed Chinese, and sometimes Turkish names, descriptions of operations, madkhals, and parameters, including those for determining true Chinese new moons. There is considerable criticism of the methods used in the Īlkhānī Zīj.

f.9r (AS 8r) **Table of the Chāgh and Kih Arguments**

A *chāgh* is a twelfth of a day, hence two hours; each chāgh is divided into eight *kih*. In this table, for each kih of the day, entries are to minutes of sexagesimal fēns.

f.9v (AS 8v) **Table of the Chinese Names of the Days of the Sexagesimal Cycle**

f.9v (AS 9r) **Table of the Chinese and Persian Names of the Days and Temperaments of the Duodecimal Cycle of Choices**

f.11v:21 (AS 11r:7) Subsection: On Determining the Solar and Lunar Equations, the True Lunar Months, and the Leap Month (*Shūn*)

Both the lunar and solar equations are defined by arithmetic rules which yield a function whose graph resembles a sine wave. However, the curve is composed of smoothly joined parabolic segments. For the moon, the period is determined by the ancient Babylonian rule equating nine anomalistic months to 248 days.

ff.12v,13r (AS 12r,v) **Table of the Chinese Year Madkhals and the Base of the Lunar Argument for the Years 781-881 Yazdigird**

For years 781, 782, 783, ..., 882 Yazdigird, 1, 2, 3, ..., 10 centuries, and individual months, entries give, precise to fēns, times of Chinese solar new year, the sexagesimal cycle of days, the interval to mean lunar new year, and the solar and lunar anomalistic arguments, or increments thereof.

ff.13v (AS 13r) **Table of the Chinese Names of the Solar Year Divisions and Related Parameters**

For each Chinese month the same entries are given as in the last table above.

f.14r (AS 13v) **Table of the Solar Equation**

The argument is 1, 2, 3, ..., 364 days; entries are fēns.

f.14r (AS 13v) **Table of the Lunar Equation**

The argument is 1, 2, 3, ..., 248 (= 9 Babylonian anomalistic months in days); entries in fēns.

f.15r **Tabular Worked Example of Yazdigird to Uighur Conversion**

Two methods are exhibited, one worked out by Kāshī.

f.15v (AS 15r) **Table of Mean Month and Division Beginnings for the Chinese Year Beginning in 778 Yazdigird**

The arguments are Chinese months and year divisions. Entries give the increase in Yazdigird days for mean month beginnings, solar and lunar anomalistic arguments, and madkhals.

f.16r:1 (AS 15v:1) Section 2. On Extracting a Chinese Date from a Hijrī Date

(On the AS film, following the exposure which shows f14v and 15r, there is a photograph of a pair of pages which seem not to belong to this manuscript, but which has been laid on it, for the left edge of AS f.16r is visible. Both pages have holes in them. The first page one the right, seems to be a version of Section 2 named above. It has two small tables. The second page, on the left, is blank, but seems to have writing on the other side.)

f.16r (Not found in AS) **A Table of Solar and Lunar Arguments for Uighur Months**

The argument is the beginnings of the Turkish months; entries are the lunar argument, the lunar equation, the compound equation, and the fēns of madkhals.

f.17r-18v (AS 16r) **Tables of True New Year Madkhals**

Arguments are Hijrī years 801, 802, 803, ..., 901; 1, 2, 3, ..., 10 centuries, and each Hijrī month. Entries are Yazdigird years, days, and fēns, and elements of the sexagesimal cycle.

f.19r (AS 17r) Section 3. Concerning Difficulties and Proofs for the Uighur Calendar

Kāshī attempts to supply explanation and theoretical underpinning for the rules and his variants by the use of a figure. However there is no evidence that a geometric model was employed by the originators of the system.

f.19r (AS 17r) **A Tabular Worked Example for the Determination of Uighur True Month Madkhals for Year 814H**

f.21v:13 (AS 19r:23) **Chapter 3. On the Malikī (Jalālī) Calendar**

In addition to the usual month names, and so on, the explanation includes the astronomical determination of year and month beginnings, placement of epagomenal days, and the use of the tables.

A separate Īlkhānī (or Khānī) era is also given. It is (f.22r:15) the vernal equinox of the year of the enthronement of Ghāzān Khān, 1 Farvardīn 224 Malikī = 13 March 1302, Julian day 2 196 685.

f.22v (AS 20r) **Table of Naw-Rūz (New Year) Madkhals and Solar True Longitudes for Years 334-1390 Malikī** (for terrestrial longitude 88° from the Fortunate Isles)

Arguments are Malikī years 1, 2, 3, ..., 33, 334, 367, ..., 1390; entries are madkhals of new year's days.

f.23r,v (AS 20v) **Chapter 4. On Determining the Festivals of All the Calendars**

There are lists of the festivals of the Malikī, Hijrī, Seleucid, and Yazdigird calendars, tables of the names of Yazdigird and Malikī days, a table of the lunar mansions for Malikī months, and the risings of Canopus for the latter.

f.24r (AS 21r) TREATISE II. ON THE DETERMINATION OF THE TRIGONOMETRIC FUNCTIONS, DECLINATIONS, AND ASCENSIONS

Introduction: Definition of Technical Terms Used in This Treatise

The list commences with definitions of the standard concepts of plane geometry, then the trigonometric functions: the chord, $\text{Crd}\, x = 2R \sin(x/2)$; sine, $\text{Sin}\, x = R \sin x$; versed sine, $\text{Vers}\, x = R(1-\cos x)$; tangent, $\text{Tan}\, x = R \tan x$; and the cotangent, $\text{Cot}\, x = R \cot x$. The latter two are defined as shadows, hence the units used for gnomon lengths are included. R is usually 60. Initial capitals of the trigonometric functions distinguish between the medieval and modern functions.

The sphere with its great and small circles and their poles are next defined in order to introduce the elements of Aristotelean cosmology with its nine spheres and their motions. The three systems of spherical coordinates are then introduced: equatorial, ecliptic (with its zodiacal signs), and horizon, then terrestrial coordinates, and finally the standard terms of spherical astronomy.

By and large, in the rest of this paper, when operations and symbols are first introduced, they are then defined. However, the so-called *Rule of Four* is frequently invoked, but undefined. Hence it is defined here.

For any spherical right triangle the acute angles are designated A and B; the legs opposite them by a and b respectively. The right angle is C and the hypotenuse c. The Rule of Four states that in any pair of such triangles with an acute angle in common ($A=A'$): $\sin a / \sin c = \sin a' / \sin c'$.

Because the functions appear as ratios, the theorem also holds for the medieval functions.

The "tangent case" of the Rule of Four, for any right spherical triangle, is

$$\text{Tan}\, A \,/\, R = \text{Tan}\, a \,/\, \text{Sin}\, b.$$

f.26r:13 (AS 22v:10) **Chapter 1. On Carrying Out Operations**
Section 1. Linear Interpolation

f.26v:3 (AS 22v:21) Section 2. On Determining the Sine and the Versed Sine (The related proofs are in II,2;2 below.)

The explanation is confined to the use of a sine table with entries for the first quadrant only. So the relations between these and the sine of arcs or angles in the other three quadrants are given.

There follows a worked example of two methods of interpolation, one "easier", the other "more precise". The first is indeed approximate, yielding precise results only if the tabulated function is linear. The second method is the linear interpolation explained in the preceding section.

The same table is used to find values of the versed sine, using the definition of the function, and the fact that the cosine of an angle is the sine of its complement.

The use of the sine to find values of the corresponding inverse functions is also explained.

f.27r:11 (AS 23r:20) Section 3. On Determining the Tangent and Cotangent (The related proofs are in II,2;2 below.)

These functions, called in the text the "first shadow" and "second shadow" respectively, are defined here as

$$\text{Tan}\, x = R\,\text{Sin}\, x \,/\, \text{Cos}\, x \quad \text{and} \quad \text{Cot}\, x = R\,\text{Cos}\, x \,/\, \text{Sin}\, x\,.$$

If R is put equal to twelve or seven, the units of the shadow length are called digits or feet respectively. Relations between the two functions involving complements and reciprocals are given.

f.27r:20 (AS 23v:3) Section 4. Determining the Declinations of Points on the Ecliptic from the Celestial Equator (The related proofs are in II,2;3 below.)

The declination, δ, of an ecliptic point is defined as its distance to the celestial equator. If the great circle arc measuring the distance is normal to the ecliptic (rather than to the equator) the distance is called the second declination, δ_2. Symmetries of the declination functions with respect to the equinoxes and solstitial points are stated.

The maximum declination (ϵ, the inclination of the ecliptic) as determined by the Īlkhānī observations, is said to be 23;30°.

To calculate a declination the text gives the rule

$$\delta = \text{arcSin}(\text{Sin}\,\lambda\ \text{Sin}\,\epsilon \,/\, R)\,,$$

where λ is the celestial longitude of the given point, and as is customary, the prefix *arc* before a function indicates its inverse.

For the second declination the rule is

$$\delta_2 = \text{arcTan}(\text{Sin}\ \lambda\ \text{Tan}\ \epsilon / R)\ .$$

Alternative rules are given for both of these determinations.

f.27v:7 (AS 23v:9) Section 5. Determining Right Ascensions (The related proofs are in II,2;3 below.)

Four rules are given for calculating right ascension, α :

$$\alpha(\lambda) = \text{arcCos}[R\ \text{Cos}\ \lambda / \text{Cos}\ \delta(\lambda)]\ ,$$

then

$$\alpha = \text{arcSin}[\text{Sin}\ \lambda\ \text{Cos}\ \epsilon / \text{Cos}\ \delta(\lambda)]\ ,$$

and

$$\alpha = \text{arcSin}[R\ \text{Tan}\ \delta / \text{Tan}\ \epsilon]\ ,$$

finally

$$\alpha = \delta_2^{-1}[\delta(\lambda)]\ ,$$

where the exponent -1 indicates an inverse second declination.

Rules are given for converting ascensions of ecliptic points in the first quadrant into ascensions in each of the other three quadrants.

f.27v:16 (AS 23v:16) Section 6. Determining the Equation of Half Daylight, Rising Amplitude, and Oblique Ascensions (Related proofs are in II,2;4.)

The symmetries of the equation of daylight function with respect to the equinoctial and solstitial points are remarked. Next is a rule for calculating the equation of half daylight, eq_d :

$$eq_d(\lambda) = \text{arcSin}[\text{Tan}\ \delta(\lambda)\ \text{Tan}\ \phi / R],$$

where ϕ is the latitude of the locality. After this is an expression for the rising amplitude, w, the distance from the east point on the local horizon to the rising point of a celestial object :

$$w(\lambda) = [R\ \text{Sin}\ \delta(\lambda) / \text{Cos}\ \phi]\ ,$$

whence

$$eq_d(\lambda) = \text{arcCos}[R\ \text{Cos}\ w(\lambda) / \text{Cos}\ \delta(\lambda)]\ .$$

A fourth rule is

$$eq_d = \text{arcSin}[w(\lambda)\ \text{Sin}\ \phi / \text{Cos}\ \delta(\lambda 0]\ .$$

As for oblique ascensions, α_ϕ , if the declination of the given ecliptic point is north,

$$\alpha_\phi = \alpha - eq_d\ ,$$

otherwise the two quantities are added.

The author notes that older writers used the parameter $\epsilon = 23;35^\circ$. Directions are given for the use of the oblique ascension tables, including localities in the southern hemisphere, and finding values of inverse oblique ascensions.

f.28v:5 (AS 24r along margin) Section 7. Determining the Terrestrial Longitudes and Latitudes of Localities

Two zero meridians are said to be in common use, they differing by ten degrees. In this zīj longitudes are reckoned from the Fortunate Isles, the Azores. See IV,1;11 below on the same topic.

f.28v:10 (AS 24r:16) **Chapter 2. Geometric Proofs of the Operations in the Preceding Chapter**

Section 1. Determination of the Sine and Versed Sine (These demonstrations are in support of II,1;2 above.)

Rule 1. On the Determination of the Chord Function

The chords of arcs of 180, 120, 72, 60, and 36° are displayed, then the sines of half of each of these arcs. For each individual computation, reference is made to the theorem of Euclid's Elements which justifies it. The section ends with the value of Sin 18° carried to eight fractional sexagesimal places.

f.29r:14 (AS 24v:13) Rule 2. Determination of the Cosine

By means of a figure, the Pythagorean Theorem is applied to obtain the relation between the cosine of an arc and its sine. A worked example gives Cos 18° to nine sexagesimal places.

f.29v:4 (AS 24v:22) Rule 3. The Half-Angle Rule for Sines

The rule named in the title is derived, again by means of a figure, and is then applied twice to obtain the sines of 9° and 15° to six fractional places each. All the intermediate results of the computations are shown.

f.30r:4 (AS 25r:18) Rule 4. Rule for the Sine of a Sum or Difference

The expressions for Sin($A\pm B$) in terms of the sines and cosines of A and B are stated, then proved by means of a figure. As usual, references are made to propositions of the Elements of Euclid justifying the various steps. As examples the calculations are shown for determining Sin 33° and Sin 3° from Sin(18°±15°).

A long gloss along the left margin of f.30r states and proves, with the aid of a figure, the expression for the sine of double an arc of known sine.

f.31r:2 (AS 26r:1) Rule 5. An Introductory Statement According to Which the Sine of

One Degree Cannot Be Made Known by a Cycle of Operations

The Persian of the ensuing demonstration is frequently imprecise. The following is an attempt to rephrase the essence of the passage. The figure consists of a circle with its horizontal and vertical diameters. On one of its quadrants, implicitly taken as the first, five equally spaced points are marked. These points define five arcs, each arc starting from the intersection of the horizontal diameter with the circle, and ending with one of the five points. The magnitude of each successive arc differs from its predecessor by a constant. The projections of the five original points on the vertical diameter determine the sines of the five arcs. It is proved that the differences between successive projections, hence the differences between successive sines, constitute a decreasing sequence.

The relevance of this demonstration to Rule 5 is not indicated.

On the margin of f.32r is a note by Kāshī, saying that he had developed a method for calculating Sin 1°, and had described it in a paper. He gives the result to ten significant sexagesimal places as

1;2,49,43,11,14,44,16,19,16 .

A second gloss, by someone else, written after Kāshī's death, says that in fact the paper was unfinished when he died.

The elegant iterative algorism he originated has been described by Aaboe in [1], but a careful examination of these passages might contribute to its history.

f.31r:20 (AS 26r:15) Rule 6. Approximate Determination of the Sine of One Degree

By successive application of the half angle rule, Sin 1;7,30° is calculated from the known Sin 9°, and Sin 0;56,15° from Sin 15°. By interpolating between these two values an approximation to Sin 1° is obtained which is precise to three fractional sexagesimal places. Kāshī goes on to cite three errors which he has uncovered in the Ilkhānī sine table which he has corrected in his.

f.32r:14 (AS 26v:18) Section 2. Determination of the Tangent and Cotangent (This is in supprt of II,1;3 above.)

Here a figure shows that the tangent of an arc is the cotangent of its complement, and, depending on the units to be taken for the gnomon length, which lines are to be taken as elements in a proportion.

f.32v:19 (AS 27r:15) Section 3. Determination of Declinations and Right Ascensions (This supplies proofs for II,1;4 and 5.)

The figure for this section illustrates the symmetries in the functions involved.

The author uses it to prove

$$\delta(\lambda) = \text{arcSin}(\text{Sin}\ \epsilon\ \text{Sin}\ \lambda / R),$$

by application of the sine theorem (referred to here as the *shikl-i mughnī*, substitution theorem), then

$$\delta_2(\lambda) = \text{arcTan}(\text{Tan}\ \epsilon\ \text{Sin}\ \lambda / R),$$

by what is called the tangent theorem (*shikl-i zillī*).

Alternative rules are

$$\delta_2 = \text{arcSin}[R\ \text{Sin}\ \delta(\lambda) / \text{Cos}\ \delta(\lambda)],$$

proved by the sine theorem, and

$$\delta_2 = \text{arcCos}[R\ \text{Cos}\ \epsilon / \text{Cos}\ \delta(\lambda)],$$

by the sine theorem.

For right ascensions, three rules are given,

$$\alpha(\lambda) = \text{arcCos}[R\ \text{Cos}\ \lambda / \text{Cos}\ \delta(\lambda)],$$

by the "first case" of the sine theorem,

$$\alpha = \text{arcSin}[\text{Cos}\ \epsilon\ \text{Sin}\ \lambda / \text{Cos}\ \delta(\lambda)],$$

by the sine theorem, and

$$\alpha = \text{arcSin}[R\ \text{Tan}\ \delta(\lambda) / \text{Tan}\ \epsilon],$$

by the tangent case of the Rule of Four.

f.34r:6 (AS 28r:4) Section 4. Determination of the Equation of Daylight and Oblique Ascensions

In this section there are two figures covering the same subject matter to illustrate variant dispositions on the celestial sphere. These are used to prove, by the tangent case of the Rule of Four, the equation of daylight

$$eq_d(\lambda,\phi) = \text{arcSin}[\text{Tan}\ \delta(\lambda)\ \text{Tan}\ \phi / R],$$

then, by the sine theorem, an expression for the rising amplitude

$$w(\lambda,\phi) = \text{arcSin}[R\ \text{Sin}\ \delta(\lambda) / \text{Cos}\ \phi],$$

also

$$eq_d = \text{arcSin}[\text{Sin}\ \phi\ \text{Sin}\ w / \text{Cos}\ \delta].$$

From the above, oblique ascensions are calculated by the rule

$$\alpha_\phi(\lambda) = \alpha(\lambda) - eq_d(\lambda,\phi).$$

There follow elaborate rules giving relations between the various functions when the given ecliptic point is in a quadrant other than the first. At one stage Kāshī gives a reference to a passage in the Almagest [35] to establish the validity of a rule.

ff.35v-39v (AS 29v-32r) **Table of the Sine Function**

There are entries for ten degrees on each page. At the head of each of the ten columns per page is an entry for Sin n for $n = 0, 1, 2, ..., 89°$. In the columns, underneath the top line, for $m = 1, 2, 3, ..., 60$ minutes of arc, entries give the increment to be added to the value at the top of the column in order to yield Sin $n;m°$. All entries are to four significant sexagesimal places.

ff.40r-41r (AS 32v-33v) **Table of the Tangent Function to 45°**

There are entries for fifteen degrees on each of the three pages. At the head of each column is an entry, $\text{Tan}_{60}n$ for $n = 0, 1, 2, ..., 44°$. In the columns, as with the sine table preceding, entries for $m = 1, 2, 3, ..., 60$ give the increment to be added to the entry at the top of the column in order to obtain Tan $n;m°$. All entries are to three significant sexagesimal places.

ff.41v-42r (AS 34r) **Table of the Tangent Function to 89;30°**

The first column of the table has, for arguments $n = 0, 1, 2, ..., 44°$, values of $\text{Tan}_{60}n$ to three fractional sexagesimal places, i.e. to sexagesimal thirds. The second column also has entries of Tan n, now for $n = 45, 46, 47, ..., 89°$, to the same degree of precision as the first. There are eleven more columns, one each for $m = 0;5, 0;10, 0;15, ..., 0;55°$. The entries in these, however, are only to seconds, not thirds. The integer parts of the entries are in sexagesimals because of the rapid increase in the tangent function as it approaches its pole at 90°. So there are entries at intervals of five minutes for arguments from 45° to and including 89;55°. Curiously, the columns for 0;5, 0;25, 0;35, and 0;55° have been left blank for arguments greater than 50° (except that the very last entry, for 89;55°, has been filled in).

ff.42v-43v (AS 34v-35v) **A Declination Table, $\epsilon = 23;30°$**

The function $\delta(\lambda)$ has been tabulated to seconds of arc at intervals of six minutes from $\lambda = 0°$ to 360°. The declination function, like the sine, is symmetrical with respect to 0° and 90°. Hence it is possible to use the same entries for four zodiacal signs. The user must remember when the arc is taken to be north and when south (or positive and negative, as we would say). The entries are carried to seconds. Each such set of entries takes a page, hence the three pages required for the table.

f.44r (AS 36r) **A Table of Second Declinations, $\epsilon = 23;30°$**

The format for the table for $\delta_2(\lambda)$ is the same as that for the first declination, except that the range of the argument is at intervals of twelve instead of six minutes. Hence there are only half as many entries, and the entire table has been compressed onto one page. Entries are still to seconds of arc.

ff.44v-72r (AS 36v-52r) **Oblique Ascension Tables**

These start with one table per page for each of the values $\phi = 0, 1, 2, ..., 47°$. At this stage Kāshī, not bothered by the time expended in computation, but perhaps worried by the amount of paper being used, compressed the tables without change of format to the point where he got two tables per page. This is the situation for $\phi = 48, 49, 50, \ldots, 61, 66;30 (=90° - \epsilon)$, and 75°. In the table for 61° all the places for entries have been left blank. In the last two tables, many places for entries are empty, presumably for values of the argument for which the function fails to exist.

For all the tables the range of the argument is $\lambda = 0, 1, 2, \ldots, 360°$, and entries are to seconds. The first table of the series, since $\lambda = 0°$, is in fact a table of right ascensions.

ff72v-74v (AS 52v-54v) A Geographical Table. A List of Cities Giving Their Longitudes from the Fortunate Isles and Their Latitudes

This gives the geographical coordinates, to minutes, of 515 localities, mostly cities, arranged according the seven "climates" of classical antiquity. The list has been published in [15] with a facsimile of the text. Its entries are also listed in [14], where their coordinates may be compared with those of some seventy-four similar geographical tables, mostly of medieval Islamic origin.

Most of the cities named are from the Near and Middle East and Central Asia, but there are many from Europe, especially the Iberian Peninsula, India, and East and Central Africa. A few cities in China are included. Not surprisingly, the table seems most closely related to those in other Persian zījes.

f.75r:1 (AS 55r:1) TREATISE III. ON THE DETERMINATION OF PLANETARY POSITIONS

Introduction: Definitions of Technical Terms Used in This Treatise

The definitions include: true and mean celestial longitude, celestial latitude, the parecliptic orb (*falak-i mumaththal*), the inclined orb, node, deferent, superior and inferior planets, epicycle, deferent and epicyclic apogees and perigees, equant, the several motions, true and mean, civil day (nychthemeron), the equation of time, anomalistic and deferent equations, solar, lunar, and planetary adjusted centers, minutes of the argument, nomenclature of the planetary latitude components, true, mean, and apparent conjunctions and oppositions, the astrological aspects, temperaments, cardines, and houses, transfer (*taḥwīl*), apparent diameters, areal and absolute eclipse digits, the latitude of visible climate, altitude, azimuth, adjusted and altitude parallax, and arcs of visibility.

f.77r:20 (AS 58v:1) Chapter 1. Determination of the True Latitudes and Longitudes of the Planets, and Operations Connected With Them

Section 1. Determination of the Equation of Time (Proofs in III,2;1 below.)

The material in this section has been described in [25] and [7], p.101. In the course of his lengthy explanation, Kāshī cites and makes use of the parameters employed in the Ilkhānī Zīj, but notes that modifications must be made because of the passage of time since the earlier zīj was written.

f.78r:10 (AS 59r:12) Section 2. Determination of the Mean Positions of the Planets at Any Time

This is an explanation of how to obtain from the mean motion tables the mean longitudes, apsidal longitudes, nodal positions and anomalistic arguments of the sun, moon, and planets for any given date and time. Also described are adjustments to be made if the longitude of the locality differs from the base locality of the zīj.

f.78v:9 (AS 59v:3) Section 3. Determination of the True Longitudes of the Seven Planets at Any Time (Proofs are in III,2;2, 3, 4, and 5 below.)

The standard Ptolemaic procedure (as set forth, for instance in [32], pp.191-202) is employed here. Commencing with the simpler model for the sun, then the moon, and finally the planets, the text explains how to obtain from the equation tables the deferent and epicyclic components to be combined, using interpolation functions, in order to end with the modified equation. This is added algebraically to the mean longitude to give the true longitude of the body in question.

f.79r:16 (AS 59v:18) Section 4. Determination of the Lunar and Planetary Latitudes (Proofs are in III,2;3, 6, 7.)

Given the argument, λ_n, the longitude of the ascending node, the lunar latitude is

$$\text{arcSin}(\text{Sin } 5° \text{ Sin } \lambda_n / R),$$

obtained from the table on folios 138v and 139r.

Calculating the latitudes of the planets, even with the help of the tables, is more complicated. The two inferior planets have three components each.

f.79v:14 (AS 60r:6) Section 5. Determination of the Planetary Sectors and Retrogradations (Proofs are in III,2;9 and 10.)

The determination of whether a planet is in direct or retrograde motion depends upon its situation with respect to its retrograde and forward stations, obtained from the table of stations. If it is beyond the retrograde station and has not yet arrived at the forward station it is retrograde. Otherwise it is in forward motion. This is related to the subject of sectors (*niṭāqāt*) studied in [19]. Background on the topic is to be found in [4].

f.80r:16 (AS 60r:18) Section 6. A Simplified Method for Determination of the Planetary True Longitudes and Latitudes (Proofs are supplied in III,2;8 below.)

The standard Ptolemaic method for determining a true longitude involves individual tables for the equations of the center and the anomaly, and a combination of the results from these by the use of an interpolation table. Kāshī substitutes for these a single table, but with two independent variables, the mean and the anomaly. In principle this is indeed a simplification, since it substitutes one operation for several. However, the two numbers to be entered will usually be between pairs of values for which the table has been computed. Then a two-way interpolation is inevitable, which is tedious and difficult.

The longitude tables have been recomputed and the results reported in [39]. The latitude tables have yet to be studied.

f.80v:5 A Note on Interpolation in These Tables

In interpolating linearly across a region which contains a local maximum, Kāshī suggests replacing one tabular entry by the sum of the maximum and the difference between the maximum and the entry being replaced. There is an analogous rule for a minimum.

f.80v:18 (AS 60v:4) Section 7. On Planetary Distances from the Center of the Universe (Proof is in III,2;11 below.)

The purpose of the determination is only to find whether, at the given time, the planet is receding from the earth or approaching it. Hence there is no need to use absolute units of distance; in all cases the deferent radius is put at 60.

f.81r:4 (AS 60v:8) Section 8. On Calculating Sequences of Planetary True Longitudes and Latitudes

For the preparation of ephemerides, sequences of planetary true positions at fixed intervals are required. This section describes an interpolation scheme which pieces together parabolic segments. The passage is translated and the procedure expressed in modern notation in [20].

f.81v:4 (AS 60v:23) Section 9. Determination of Half Daylight (Proofs are in III,2;12 below.)

Kāshī uses here a function he calls "right ascension beginning from the first point of Capricorn", or "...at the Cupola (*qubba*)". This function is now known as "normed right ascension" (in [33], p.42). It is defined as

$$\alpha'(\lambda) = \alpha(\lambda) + 90^\circ .$$

It is useful because of its property that

$$\alpha'(M) = \alpha_\phi(H) ,$$

where M is upper midheaven, the ecliptic point intersecting with the local meridian, and H is the ascendent.

From this it follows that half the length of daylight is

$$\alpha'(\lambda) - \alpha_\phi(\lambda) ,$$

provided that λ is the solar true longitude at the time in question. The section gives this rule.

f.81v:13 (AS 61r:3) Section 10. Determination of Conjunctions and Oppositions

Since an eclipse is possible only at an apparent syzygy, it is important that the time of the syzygy be calculated with maximum precision. This is the subject of the section. With an ephemeris at hand it is easy to determine that a conjunction or opposition will occur sometime between a pair of successive noons. Knowing the true longitudes of sun and moon at each of the two noons, and assuming the angular velocities of both luminaries to be constant during the twenty-four hours in between, t, the time in hours from the first noon to the conjunction will be

$$t = 24d/b ,$$

where d is the elongation between the two luminaries at the first noon, and b is the rate of elongation in degrees per day.

There is a table (beginning at f.58v) in the zīj for solving the problem in this manner, and in [18], pp. 79, 241-243, Kāshī describes a device he has invented for solving it mechanically. Variants of the table are in many zījes.

This solution is approximate. For a more precise determination Kāshī prescribes an elaborate procedure which begins with calculations of the true longitudes of the sun and moon at the beginning and end of the hour during which the conjunction takes place.

f.83r:20 (AS 65r:4) Section 11. Determination of the Ascendent at a Conjunction or an Opposition

The time of the syzygy having been determined in the preceding section, to find the longitude of the ecliptic point rising at that instant, multiply by fifteen the time in hours from the preceding noon to the conjunction. Add the result to the normed right ascension (defined in Section 9 above) of the sun at the time of the conjunction. The inverse oblique ascension of the result is the required ascendent. Symbolically it is

$$\alpha_\phi^{-1}[\alpha^\circ(t+15h)] .$$

To this rule Kāshī adds suggestions for improving the result.

f.83v:10 (AS 62v:1) Section 12. Determination of a Lunar Eclipse by Computation (Proofs are in III,2;13 below.)

With the time of the opposition in hand, a test of distance from the node is applied to see if an eclipse is possible. If it is, the time of the middle of the eclipse is then calculated, as well as the apparent diameters of the moon and the shadow cone. From them the magnitude and duration are determined.

f.84v:20 (AS 62v:14) Section 13. Determination of a Lunar Eclipse by the Use of Tables

The operations required, facilitated by the use of tables, are described in III,2;13, at f.116r:8.

f.85r:10 (AS 62v:22) Section 14. Determination of a Solar Eclipse by Computation (Proofs are in III,2;14 below at f.119r:11.)

After carrying out a preliminary test to make sure an eclipse is possible, the rest of the procedure is so lengthy that it has been divided into the following six steps:

f.85r:15 (AS 63r:1) Introduction 1. Determination of the Latitude of Visible Climate (the angle the ecliptic makes with the local horizon, for which three methods are given.)

f.85v:11 (AS 63r:6) Introduction 2. Determination of a Luminary's True Altitude (as distinguished from its apparent altitude, which is affected by parallax.)

f.86r:16 (AS 63r:24) Introduction 3. Determination of Adjusted Lunar Parallax and Apparent Zenith Distance

f.86v:6 (AS 63v:16) Introduction 4. Determination of the Apparent Lunar Longitude and Latitude (by calculating and allowing for the longitude and latitude components of parallax.)

f.87v:14 (AS 64r:18) Introduction 5. Determination of the Time of Apparent Conjunction

f.88r:13 (AS 64v:5) Introduction 6. Determination of the Arcs Subtended by the Apparent Radii of Sun and Moon at the Time of the Eclipse (This culminates with calculation of the eclipse magnitude, and the durations of immersion and totality.)

f.89r:8 Section 15. On the Determination of Solar Eclipses by the Use of Tables

Here again the operations outlined in the preceding section are facilitated by tables.

f.89v:1 (AS 65r:13) Section 16. On the Determination of Lunar Crescent Visibility, and the Appearances and Disappearances of the Planets

The Muslim calendar is purely lunar and, as with the Jewish calendar, the beginning of a month is in principle determined by the first sighting of the new crescent. Hence it is of interest to investigate beforehand whether such a sighting is possible on a particular evening.

For this purpose the adjusted elongation between the two luminaries at the time of sunset is calculated. From this it can be predicted that either the crescent will not be seen, or will barely be visible, or plainly visible.

Because of its connection with the end of the fasting month, crescent visibility theory is a favorite topic with Muslim astronomers, and many, but undoubtedly not all of their efforts have been studied and published. A recent paper on the subject by Doggett and Schaefer, [9], not only reports the observations of many persons spotted all over the United States, but uses their results to work out criteria for visibility. Their bibliography also lists most of the modern descriptions of medieval work.

The calculation of first and last visibility of the planets is far more complicated, and no attempt is made to outline it here (but see [12]).

f.90r:19 (AS 65v:18) Section 17. On Determining the Time of Arrival of the Sun at a Given Position

The solution of this problem is of astrological interest, in particular the year-transfer (*tahwīl al-sina*), determination of the instant of the vernal equinox.

A pair of successive noons is found such that at the first the sun has not yet arrived at the given point, and at the second it has passed it. The time of arrival in hours is then $24d/r$, where d is the distance from the first noon position to the desired one, and r is the solar travel in degrees per day.

This is approximate. For situations where more precision is required, an elaborate computation is described, commencing from mean and apsidal positions.

f.91r:4 (AS 66r:13) Section 18. On Determining Conjunctions of the Moon With the Planets

The approach here seems to be essentially that of Section 10 above.

f.91r:16 (AS 66r:21) Section 19. On Determining the Aspects of the Planets

A pair of planets is said to be in such and such an astrological aspect when their longitudes differ by a fixed number of zodiacal signs depending upon the particular aspect: *sextile* is two signs, *quartile* is three, *opposition* is six (See [2], p.225). The method of finding the time of arrival at an aspect also resembles that of Section 10 above.

f.91r:23 (AS 66r:23) Section 20. On the Equalization of the Astrological Houses (Proofs are in III,2;17 below.)

The houses are twelve divisions of the ecliptic which, unlike the zodiacal signs, vary depending upon the time and the locality for which they are calculated. The initial points of four of the houses are the intersections of the ecliptic with the local horizon and the meridian. For the determination of the remaining eight cusps, as they are called, there are a number of different methods. Kāshī gives three of these, which are in turn described in [28].

f.92r:3 (AS 66v:16,15?) Section 21. Determination of the Temporal (Unequal) Hours

An unequal hour is a twelfth of the length of daylight or night, hence its length depends on the season and the local latitude. Kāshī gives rules for conversions between times in equal and unequal hours.

f.92r:16 (AS 66v:22) Section 22. Determination of the Hours of Sanctuary (? *bast*)

This section describes an astrological cycle which commences from the time of a conjunction and runs through the seven planets in succession until the next conjunction, assigning twelve unequal hours to each planet.

f.92v:9 (AS 67r:5) Section 23. On Determining the Duration of the Effects of Solar and Lunar Eclipses

Again purely astrological, this section converts the time and duration of an eclipse into longer spans through which its effects will last.

f.92v:19 (AS 67r:10) Section 24. On the Positions of Some of the Fixed Stars

The author remarks that since in the copies and translations of the Almagest there are differences in the coordinates of the fixed stars, and in the Īlkhānī observations not all the stars were investigated, further observations are to be hoped for. (These were indeed carried out by Kāshī and others at Samarqand.) The table in the zīj is for the beginning of 801 Yazdigird.

f.93r:10 (AS 67v:1) **Chapter 2. Geometric Proofs of the Operations in This Treatise**

Section 1. The Equation of Time (Derivation of III,1;1 above. See [25].)

This is a detailed explanation of how to calculate the difference between true and apparent solar time. In the zīj, figures are used to good effect, and numerical parameters embedded in the tables are cited.

f.95r:13 (AS 68v:8) Section 2. The Solar True Longitude (Relevant to III,1;3.)

With the aid of a figure, the Ptolemaic eccentric model for the solar motion is explained rigorously and in detail. The eccentricity, said to have been determined from new observations, is taken as 2;6,9 units of R, the deferent radius. Geometric operations are justified by reference to the relevant propositions in Euclid's Elements.

f.95v:18 (AS 68v:25) Section 3. The Lunar True Longitude and Latitude (Relevant to III,1;3.)

The much more complicated Ptolemaic model for the moon is given like treatment. For the eccentricity, the author accepts Ptolemy's 10;19, but for the epicycle radius he takes 5;16,47, rounded off from the results of his own lunar eclipse observations reported in the preface to the zīj. This value replaces the Ptolemaic 5;15. To buttress an operation he cites a proposition from Menelaos' Spherics. His figure for deriving Ptolemy's interpolation table shows the epicycle in the three traditional positions of minimum, maximum, and intermediate distance.

It is of great interest that in criticizing Tūsī's presentation, Kāshī makes specific mention of the lunar model in the *Tadhkira* [37]. This mechanism, taken over by Copernicus, made a radical improvement over that of Ptolemy in the matter of lunar distance from the earth. Nevertheless, Kāshī retains the Ptolemaic version.

f.98r:20 (AS 70v:6) Section 4. The True Longitude of the Superior Planets and Venus (Relevant to III,1;3 above.)

In this section the Ptolemaic model for the motion of these four planets is meticulously derived, with tables showing the results of intermediate computations.

As for parameters, the author accepts Ptolemy's eccentricities of 3;25, 2;45, and 6;0 for Saturn, Jupiter, and Mars respectively, but for Venus he adopts 1;3, based on new observations, he says, instead of the Ptolemaic 1;15. For epicycle radii he takes the Ptolemaic 6;30, 11;30, and 43;10 for Saturn, Jupiter, and Venus respectively, but for Mars 40;18 (or 40;48).

f.99v:18 (AS 71r:24) Section 5. The True Longitude of Mercury (Relevant to III,1;3 above.)

For this planet the derivation is based on the Ptolemaic model and parameters.

f.100v:11 (AS 71v:18) Section 6. Latitudes of the Superior Planets (Relevant to III,1;4 above.)

No numerical computations appear in this section.

f.101v:19 (AS 72v:4) Section 7. Latitudes of the Inferior Planets (Relevant to III,1;4.)

The three latitude components of each of these two planets require long explanations, again without actual computation being shown. The differing results of Ptolemaic and more recent observations are cited.

f.104v:6 (AS 74r:19) Section 8. Determination of Planetary Longitudes and Latitudes by a Method Originated by the Author (Relevant to III,1;8.)

This is Kāshī's derivation of his own method of computing planetary longitudes and latitudes. It deals with all five planets simultaneously. Only for Venus are the lengthy numerical calculation displayed.

f.108v:20 (AS 77r:15) Section 9. Determination of Retrogradations and Forward Motions (Relevant to III,1;5.)

The theorem of Apollonius for locating a retrograde station is cited and applied. The Īlkhānī observations led to parameters for Mars and Venus which differ from those of the Almagest. Hence two sets of tables are presented, one from the Īlkhānī zīj, the other from the Almagest.

f.113r:15 (AS 80r:6) Section 10. Planetary Sectors (Supplementary to III,1;5 above. See [19], pp. 247-253.)

Definitions of the sectors, deferent and epicyclic, are given. There are directions, with proof, for calculating. These are in general terms, with no particular planet named, and no actual calculations displayed.

f.114v:16 (AS 81r:11) Section 11. Planetary Distances from the Center of the Universe (Suppl. to III,1;5.)

This is a brief statement concerning the layout of the distance tables. Kāshī alludes to a proof already given. Perhaps he was thinking of a tract he wrote called *Sillum al-Samā'* (The Heavenly Ladder) on planetary distances. He refers specifically to it two sections later.

f.115r:1 (AS 81r:18) Section 12. Determination of the Length of Half Daylight, and of the Ascendent from the Time of Day (Cf. III,1;9 above.)

A formal proof is given only for the determination of the ascendent.

f.116r:8 (AS 82r:3) Section 13. Determination of Lunar Eclipses (Supplementary to III,1;12.)

This is a detailed exposition of the conditions to be satisfied if the body of the moon is to enter the shadow cone cast by the earth. Numerical results are compared with those obtained by Kāshī himself in his work *Sillum al-Samā'*.

f.119r:11 (AS 84r:4) Section 14. Operations Connected With Solar Eclipses (Supplementary to III,1;14 above.)

The section first derives and proves three rules for calculating the latitude of visible climate. Next is the determination of the altitude of a luminary in terms of its coordinates, the longitude of the ascendent, and the latitude of visible climate. There follow expressions for lunar parallax in altitude, longitude and latitude components of this, whence the lunar apparent position, and finally the derivation of eclipse limits.

f.123v:19 (AS 87r:1) Section 15. Determining Planetary Appearances and Disappearances (Supplementary to III,1;16.)

Commencing with the moon, Kāshī accepts without demonstration that a basic parameter for crescent visibility is ten time degrees in setting between the sun and the moon. This is to be modified by what he calls the equation of setting, the effect of the moon's considerable latitude. For this he gives a detailed exposition of how to calculate it.

He then switches to the planets, giving from Almagest 13,7 the solar depressions after sunset or before sunrise required to insure visibility. These must be converted for each of the five planets into longitude differences, affected not only by the latitude of the locality, but also by the celestial latitude of the several planets. Explanation of these takes up the rest of the section.

f.124v:12 (AS 87v:3) Section 16. Determination of the Sun's Time of Arrival at a Given Position (Cf. III,1;17 above.)

For a precise solution, the geometric configuration is somewhat complicated. Once the procedure is carried through, however, the rest is a matter of successive approximations until the desired accuracy is attained.

f.125r:6 (AS 87v;14) Section 17. Equalization of the Astrological Houses (Cf. III,1;20, also [28].)

Of the three methods of equalization presented in Chapter 1, Kāshī here gives full treatment to the first two: figures, explanation, and proof. They are the Standard

Method and the Prime Vertical Method, rightly attributed to Bīrūnī, and called by Iranian zīj writers the "method of established centers". The third, the Dual Longitude Method, attributed to the Maghribis by the Iranians, is not mentioned here. Perhaps Kāshī deemed it sufficiently straightforward to demand no proof.

ff.126v,127r (AS 89v,90r) **Table of the Equation of Time** (Partially recomputed in [25].)

The argument is 0, 1, 2, ..., 360° of solar longitude. There are three entries for each of these, all to seconds. The first is the equation of time as of the beginning of 712 Yazdigird (= 21 December 1342). The second is the correction to be added algebraically a century after epoch, and the third the correction for seven centuries. The maximum error for those entries which have been recomputed is a second.

ff.127v-130r (AS 92r) **Mean Motion Tables**

Arguments are years 781, 782, 783, ..., 791 Yazdigird (giving positions), and 10, 20, 30, ..., 100, 200, 300, ..., 1000 years, 1, 2, 3, ..., 12 months, 0, 1, 2, ..., 30 days, 1, 2, 3, ..., 60 hours (giving motions). All entries are to three fractional sexagesimal places, means, anomalies, and nodes, of the sun, moon, and the five planets.

There is a separate table of apsidal positions, to seconds, for years 781, 782, 783, ..., 790. (In the Aya Sofya film only two pages appear which have mean motions. One is 92r. The other is opposite 92r, but upside down, and with the main title not in the photograph. It is probably a copy of IO 128v.)

ff.130v-131v (AS 91r,92v, 93r) **Solar Equation Table**

The argument is 1, 2, 3, ..., 360°. Entries are to three fractional sexagesimal places. The maximum value is 2;0,29,10° at 92°.

f.132r (AS 93v) **A Table Giving the Equation of Time Correction for the Solar Mean**

Argument: 0, 1, 2, ..., 360° of solar true longitude; entries are to seconds.

f.132v (AS 93v) **Table of the First Lunar Equation and the Related Interpolation Function**

Arguments: 0, 1, 2, ..., 360°; entries are to seconds. For the equation, entries are in effect identical with those of the Almagest. But for the interpolation function there is a systematic divergence from the Almagest.

f.133r (AS 94r) **Table of the Second Lunar Equation** (epicyclic) **and Its Increase at Minimum Distance**

Argument: 0, 1, 2, ..., 360°; entries to seconds. The maximum equation is 5;2,53° at 95°.

(The next six tables are missing from AS.)

f.133v **Table of the Equation of Time Correction for the Moon**

Argument: 0, 1, 2, ..., 360°; entries to seconds.

Table of the Third Lunar Equation

Argument: 0, 1, 2, ..., 360°; entries to seconds. This is the change in the moon's longitude due to its latitude.

f.134r **Tables of the First and Second Equations of Saturn**

Two blocks of tables, each block tabulating two functions, have been squeezed onto a single page. For all the arguments the domain is 0, 1, 2, ..., 360°. The first block has the first equation and the interpolation function. The second has the difference in the epicyclic equation between minimum and maximum distances of the epicycle. Side by side with this is the second equation. All the entries are carried to minutes.

f.134v,135r **Table of the First and Second Equations of Jupiter**

This has the same format as the tables for Saturn, except that now, and for the equation tables of all the remaining planets, each block is on a separate page. The column for the interpolation function precedes the first equation.

f.135v,136r **Table of the First and Second Equations of Mars**

For this planet the interpolation function has been carried to seconds, and the difference column precedes the second equation.

f.136v,137r **Table of the First and Second Equations of Venus**

Here the order of the functions is the same as that for Jupiter.

f.137v,138r **Table of the First and Second Equations of Mercury**

For this planet the equation function is tabulated last on both pages.

ff.138v,139r (AS 97v) **Table of Lunar Latitude**

The range of the argument is 0, 0;12, 0;24, ..., 360°. Entries are to seconds.

Maximum is 5°.

f.139r (AS 98r) **Table of the Interpolation Functions for the Latitudes of the Superior Planets**

The range of the arguments is 0, 1, 2, ..., 360°. All entries consist of single sexagesimal digits. There is one table for both Saturn and Jupiter, so arranged that the same entries are used for both planets, but with different values of the arguments. A second table is for Mars.

f.139v (AS 98r) **Table of the Latitudes of the Superior Planets**

The range of the arguments is 0, 1, 2, ..., 360°. For Saturn, Jupiter, and Mars there are entries, to minutes, for a northern and a southern function each.

(AS 98v) **Table of the First Latitude of Venus and Its Interpolation Functions**

The range of the arguments is 0, 1, 2, ..., 360°. Three functions are tabulated, to one sexagesimal digit each. The first is the first latitude of Venus, with a maximum entry of 10. The other two are typical interpolation functions ranging from zero to sixty.

f.140r (AS 98v) **Table of the Second (*mayl*) and Third (*inhirāf*) Components of the Latitude of Venus**

The domain of the arguments of both functions is 0, 1, 2, ..., 360. Entries are calculated to minutes.

f.140v (AS 99r) **Table of the First Latitude of Mercury and Its Interpolation Functions**

The layout and precision of this table is the same as the analogous one for Venus above, except this has twice as many columns. Here the maximum entry for the first latitude is 45.

f.141r (AS 99r) **Table of the Second and Third Components of the Latitude of Mercury**

The layout and precision of this table is identical with that of the analogous table for Venus above.

f.141v (AS 99v) **Table of Retrograde and Forward Stations of the Five Planets** (Cf. III,1;5 above.)

Argument: 0, 6, 12, ..., 360°; entries to minutes.

f.141v (AS 99v) **Table of Planetary Sectors** (Cf. III,1;5.)

These consist of deferent distance and velocity sectors, and distance sectors of the sun (where they exist), the moon, and he five planets. Entries are usually to minutes, occasionally to seconds.

f.141v (AS 99v) **Table of Maximum and Minimum Durations of Planetary Retrogradations and Forward Motions** (Cf.III,1;5.)

Entries are in days and hours for each of the five planets.

f.142r (AS 100r) **Table for Simplified Determination of the Solar True Longitude**

Argument: 0, 1, 2, ..., 360°. For this and all the simplified longitude tables which follow, entries are true longitudes from the apogee, here to seconds.

ff.142v-144r (AS 100v-102r) **Table for Simplified Determination of the Lunar True Longitude**

For the mean longitude and the anomaly the domains are 0, 5, 10, ..., 360°. Entries are to minutes here, and in all of the simplified tables which follow.

ff.144v-148r (AS 104v-106r) **Tables for Simplified Determination of the True Longitudes of the Superior Planets**

Tabular forms have been ruled in for all three planets, but only for Jupiter have the entries been filled in. For it the domain of the mean longitude is 0, 5, 10, ..., 360°; that of the anomaly 0, 15, 30, ..., 360°.

ff.148v-150r **Tables for Simplified Determination of the True Longitude of Venus**

For the first and fourth quadrants the interval between values of the anomalistic argument is ten degrees. In the two middle quadrants, many intervals are less than ten, but irregular. Mean longitude: 0, 5, 10, ..., 360. (In the Aya Sofya film ff.106v-111r are missing.)

ff.150v-151v **Tables for the Simplified Determination of the True Longitude of Mercury**

Forms have been ruled for the table, but only the title has been written in. The rest is blank.

ff.152r-156r (AS 111v-113r) **Tables for Simplified Determination of the Latitudes of the Five Planets**

Forms have been ruled in, one page each, for Saturn, Jupiter, and Mars, but there are no entries for either manuscript.

For Venus the form takes up three pages, but only the first, f.153v, is completely filled with entries. The next page, f.154r, has been partially filled in, but rectangular regions have been left blank, bordered by filled in rows and columns. The last page, f.154v, is even more sparsely filled in than the previous one. It consists entirely of empty rectangles.

In the Aya Sofya version the first page of this table is missing. The second and third, 112r and 112v, are identical with the corresponding pages of the India Office version.

This does not look like a case of scribal omissions. Apparently Kāshī calculated true longitudes along certain rows and columns, leaving the rectangular spaces to be filled in by interpolation. This he never got around to completing.

The domain of both the mean longitude and the anomalistic argument in the Venus table is 0, 10, 20, ..., 360°.

There is a table for Mercury in which the domain of the anomaly is the same as that of Venus, but that of the mean longitude is 0, 6, 12, ..., 360°. Here all the entries have been filled in. However, AS 113r has only the empty ruled form; there are no entries. This may have been left out by a scribe. It is also possible that the Aya Sofya version was compiled before that of the India Office, and that the Mercury table was computed in the interim.

f.156v (AS 114v) **Table of the Distances of the Planets from the Earth**

There are tables for the moon and the five planets. For all, the single argument is the adjusted anomaly: 0, 5, 10, ..., 360°. For the moon, the entries for the equation are to two significant sexagesimal places. Units of the entries take a sixtieth of the deferent radius as one for all the tables. Two functions are tabulated: the greatest distance, and the equation, the amount to be subtracted, depending on the anomaly. For the moon, entries for the equation are to two significant sexagesimal places, whereas entries for the greatest distance are to three places. For all the rest, entries for both functions are to two significant places.

f.157r (AS 115r) **Table of the Solar Distance from the Earth and Tables of Interpolation Functions for the Planets**

For all the functions, the argument is longitude measured from apogee: 0, 5, 10, ..., 360°. For the sun only, it having only one equation, the unit for the entries is a sixtieth of the solar deferent radius, carried to two fractional sexagesimal places.

In the case of the moon and the five planets, they having two equations, their distances from the earth depend on two independent variables. Hence an interpolation function for each is required in order to calculate the effect on the distance caused by the second equation. Entries for these functions are provided in six adjacent columns. In all cases the units are sixtieths of the respective deferent radii. For the moon only, entries are carried to one fractional sexagesimal place. For all the planets, the entries are integers.

These interpolation functions are to be used with the table on the preceding page in calculating planetary distances.

Table for Simplifying Interpolation

Arguments are $n = 1, 2, 3, ..., 6$ and $m = 1, 2, 3, ..., 30$. Entries are $nm/6$ rounded to an integer. This table is to be used in conjunction with the two preceding sets of tables for determining planetary distances.

ff.157v,158r (AS 115v,116r) **A Table of Right Ascensions Reckoned from Capricorn**

These are normed right ascensions (the function defined in the commentary above to 81v:4) for longitudes 270, 271, 272, ..., 269°; entries are carried to seconds.

ff.158v-162r (AS 116v-120r) **Tables of the Hours Required for the Sun or Moon to Traverse Given Minutes of Arc**

There are two tables, one for the sun, the other for the moon. For both, one argument is $m = 1, 2, 3, ..., 10, 20, 30, 50$ minutes of arc. The second is the rate of advance in minutes per hour. For the sun it is 2;24, 2;25, 2;26,... , 2;34. For the moon 23;45, 23;50, 23;55, ..., 42;25. Entries are m/n, the number of hours required for the object to traverse m minutes of arc, carried to seconds of time.

f.162v (AS 120v) **Table of Conjunctions and Oppositions**

For the beginnings of Hijrī years 801, 802, 803, ..., 811 the date in the Yazdigird calendar is given, followed by the time of day, the solar mean longitude measured from the apogee, these to three fractional sexagesimal places; the lunar argument, and the position of the node, to seconds of arc.

The motions of the same quantities, to the same degree of precision, are entered for 10, 20, 30, ..., 100, 200, 300, ..., 1000 Hijrī years, and 1, 2, 3, ..., 12 Hijrī months.

f.163r (AS 121r) **Table of an Hour's Solar and Lunar Travel in Conjunctions and Oppositions**

For values of the solar or lunar anomaly of 0, 5, 10, ..., 360°, the hourly travel of the sun, and the hourly travel of the moon is given both on the ecliptic and from its node, all to seconds of arc.

f.163r (AS 121r) **Table of Lunar Disk and Shadow Radii During Conjunctions and Oppositions**

For values of the adjusted lunar anomaly of 0, 5, 10, ..., 360° the radius of the lunar disk, both for solar and lunar eclipses, and the shadow radius (for the latter, a column each for solar longitudes from the apogee of 1, 2, 3, ..., 6 zodiacal signs); all entries are to seconds of arc.

f.163r (AS 121r) **Table of Adjusted Lunar Parallax in Altitude**

For lunar zenith distances of 0, 2, 4, ..., 90°, entries to seconds give the lunar parallax and its equation. For modifying the equation there is an interpolation function, to seconds, the argument of which is 0, 5, 10, ..., 360° of the adjusted lunar anomaly.

ff.163v,164r (AS 121v,122r) **Lunar Eclipse Table**

For lunar daily rates of 11;50, 12;10, 12;30, ..., 14;50°, at lunar latitudes of 0, 1, 2, ..., 70 minutes of arc, four functions are tabulated: linear and areal digits, and times of totality and immersion, all to minutes.

ff.164v,165r (AS 122v,123r) **Tables of Intervals Between True and Apparent Conjunction and Lunar Parallax in Latitude**

Four tabular forms have been ruled, for localities of latitude ϕ = 20, 30, 40, and 50°, but of these only the one for 30° contains entries. For noon, and 1, 2, 3, 4, 5, 6, and 6;57 hours before noon and after noon, and for each zodiacal sign, there are two entries: the number of hours between true and apparent conjunction, and the lunar parallax in latitude, both to minutes.

f.165v (AS 123v) **Solar Eclipse Table**

For lunar daily rates of 11;50, 12;10, 12;30, ..., 14;50°, and for lunar latitudes of 0, 2, 4, ..., 34 minutes, three functions were to have been tabulated: linear and areal digits, and times of immersion. However, the column for areal digits has been left blank. The other two have been carried to minutes.

f.166r (AS 124r) **Visibility Tables for the Moon and the Five Planets**

For the moon there is a table of the equation of setting, to minutes, for the second, third, fourth, and fifth climates, for lunar latitudes of 1, 2, 3, 4, 5°, and for each zodiacal sign.

For the five planets there are tables of the arc of visibility, to minutes, for the third and fourth climates. For the superior planets there are two phenomena: evening disappearance before reaching the sun, and morning appearance after passing it. For the inferior planets, in addition to the two events just mentioned, there are also morning disappearance before passing the sun and evening appearance after passing it.

f.166v (AS 164r) **Table of the Center of Transfer** (*markaz-i tahwīl*)

This is the function $\lambda_a + 2;0,29 \sin \lambda_a$, where $\lambda_a = 0, 1, 2, ..., 360°$, is longitude measured from the apogee. The entries, carried to seconds, are approximations to the solar mean longitude measured from the apogee.

(AS 125r) **Table of Precessional Motion**

There are entries, to thirds, for 1, 2, 3, ..., 10, 20, 30, ..., 100, 200, 300, ..., 1000 Yazdigird years, and 1, 2, 3, ..., 12 months.

f.167r (AS 125r) **A Star Table**

Eighty-four stars are listed. For each, longitude and latitude are given, to minutes, magnitude, and temperament (*majāz*, pl. *amzaja, mazajāt*, association with planets, [33], p.954). The epoch being 801 Yazdigird, longitudes have been increased 19;36° over those of the Almagest.

Professor Paul Kunitzsch, having examined the table, kindly reports that its epoch is 29 November, 1431. He explains Kāshī's precessional increase in the longitudes over that of the Almagest as follows. He noticed that Kāshī's star names are practically identical with those of Ṭūsī's Īlkhānī Zīj. This is not surprising, since the Khāqānī zīj is supposed to be an improved version of the latter. Apparently Kāshī accepted Ṭūsī's corrections to the Īlkhānī star table, based on a precessional rate of a degree in 66 years. To these he added an additional correction, this using a rate of a degree per 70 years.

f.167v (AS 125r) TREATISE IV. VARIOUS OPERATIONS OF SPHERICAL ASTRONOMY (This treatise has been described in [24].)

Introduction: A Technical Glossary

Among the terms defined are: distance to the equator, ascension of transit, degree of transit, maximum altitude, mean sine, adjusted diameter (or hypotenuse), rising and setting ascensions, degree of rising, argument of azimuth, azimuth, equation of azimuth, distance between stars, east-west line, and inclination (or deviation) of the *qibla*.

f.167v:24 (AS 126r:6) **Chapter 1. Rules for the Operations**

Section 1. Determination of Stellar Declinations

The rule is a complicated sequence commencing with the determination of a quantity

$$f = \text{arcSin}[\text{Sin } \lambda \text{ Cos } \beta / R],$$

where β is celestial latitude. Kāshī then gives three similar expressions, the last of which is the required declination, δ.

There follows an alternative rule, of comparable length.

f.168r:14 (AS 126r:22) Section 2. Determination of the Ascension of Transit

This amounts to right ascension in the modern sense. A first rule is

$$\alpha = \text{arcCos}[(\text{Sin} f / \text{Cos } \delta) R],$$

where f is the quantity found in the preceding section. Three alternative rules follow.

f.168v:10 (AS 126v:22) Section 3. Determining the Culminating Altitude of a Star

The rules of this section follow immediately from the equivalent of the expression

$$h_{max} = (90° - \phi) + \delta,$$

where h stands for altitude.

f.168v:21 (AS 127r:2) Section 4. Determination of the Mean Sine

This may be defined as

$$\text{mSin} = ½[\text{Sin}(\phi+\delta)+\text{Sin}(\phi-\delta)].$$

A number of equivalent expressions are proved, such as

$$\text{mSin} = \text{Cos } \delta \text{ Cos } \phi / R.$$

The mean sine is applied below in determining the culminating altitude of a star.

f.169r:5 (AS 127r:12) Section 5. Determination of the Equation of Half Daylight and the Length of Half Daylight (d)

Half daylight is the algebraic sum of a quadrant plus the equation of half daylight, rules for which were derived above. Here the author proves that

$$d = \text{arcVers}[\text{Sin } h_{max} / \text{mSin}],$$

together with other related expressions.

f.169v:17 (AS 127r:24) Section 6. Determining the Ascendent and Descendent of a Star

The rules obtained here involve manipulations with the oblique ascension functions.

f.169v:4 (AS 127v:10) Section 7. Conversion from Altitude to Azimuth

The section begins with the equivalent of the expression

$$\text{Sin } h \text{ Sin } \phi / \text{Cos } \phi = \text{Sin } h \text{ Tan } \phi / R$$

for calculating the "share of the azimuth". There follow a series of derivations involving the rising amplitude and culminating in the desired azimuth. The method is of Indian origin.

f.170r:5 (AS 128r:11) Section 8. Determination of Altitude from Azimuth

Only one method is given, which is at least as old as Bīrūnī. It gives the expression

$$\text{arcSin}[\text{Cos } az \text{ Cos } \phi / R]\ ,$$

to produce an auxiliary quantity, which is used to turn out a second, the second a third. The algebraic sum of the second and third auxiliary quantities is the desired altitude.

f.170r:17 (AS 128r:18) Section 9. Determination of Oblique Ascensions for a Given Locality

The standard method for calculating oblique ascensions has already been given in III,1;7 and III,2;4 above. Kāshī has thus been presented with an opportunity to exhibit his versatility, which he does by expounding a completely different method using the latitude of visible climate, normed right ascensions, and the tangent function. It is much longer than the usual method.

f.170v:7 (AS 128v:6) Section 10. Determination of Inverse Ascensions

The section begins with two algorisms for finding the inverse right ascension of a given arc, one employing second declinations. Kāshī then turns to the problem of inverting oblique ascensions and solves it by using inverse right ascensions.

f.171r:2 (AS 128v:24) Section 11. Determination of Terrestrial Coordinates

Three ways are given to find the local latitude. One is: observe the altitude of the sun at the two solstices. Then

$$\epsilon = \tfrac{1}{2}(h_{max}-h_{min})\ ,$$

whence

$$\phi = h_{min} + \epsilon\ .$$

For the much more difficult problem of longitudes Kāshī has two solutions. One is to chose a locality of known longitude, and arrange that a lunar eclipse be observed from that place and also from the locality of unknown longitude. Then the time difference between, say, the middle of the eclipse at the two places, when converted into degrees, is the required longitudinal difference between the two places. This technique had previously been applied (see [3], p.164).

The second method assumes as known the latitudes of the two localities and the great circle distance between them. The Pythagorean theorem is then applied to give the longitude difference. It is astonishing that anyone with Kāshī's intimate knowledge of the geometry of the sphere would propound such a scheme. It contrasts with Bīrūnī's admirable solution of the same problem in [3].

f.171v:9 (AS 129v:6) Section 12. Determination of the Distance Between Two Stars

After disposing of special cases, the general problem is attacked. It is solved by the successive determination of four quantities which are parts of various spherical triangles.

On the same page is a long marginal gloss by Ulugh Beg himself discussing aspects of the problem. This page is reproduced in facsimile in [30],which also describes the contents of the gloss. In it Ulugh Beg remarks, correctly, that the section can be improved by adding to it one more special case. This he does, adding to the rule a proof of it with the aid of a figure. Thus the gloss supplies strong evidence that the astronomer prince was a highly competent mathematician and astronomer as well as an administrator (see also [26]). At the corresponding place, p.205, in the Cairo copy the only thing in the margin is a copy of the statement in the India Office manuscript that the sultan has added a gloss, but the actual gloss does not appear.

f.172v:2 (AS 130r:22) Section 13. Determination of the Local Meridian

Two methods are described. One is to observe the altitude of a celestial body of known coordinates, and simultaneously to mark the direction of the shadow it causes a gnomon to cast. By methods previously presented, calculate the azimuth of the shadow from the altitude. From the azimuth lay off the cardinal directions.

Alternatively calculate beforehand for a particular day the altitude for which the sun will have zero altitude. The shadow direction at that time will be the east-west line.

f.172v:23 (AS 130v:19) Section 14. Determining the Angle Between the Meridian and Any Horizontal Line

Assuming the two to be in the same plane, from a point on the line draw a perpendicular to the meridian. Then the length of this perpendicular, in units such that the hypotenuse of the resulting triangle is sixty, is the (medieval) sine of the required angle.

f.173r:7 (AS 130v:25) Section 15. Determination of the Qibla

The Muslim, when making his five daily prayers, faces toward Mecca. So it is of practical importance that he know the angle between his meridian and the great circle connecting his station with Mecca. Hence solutions of this problem of the qibla, as it is called, proliferated in the world of Islam.

Kāshī disposes of several special cases. For the general situation he uses Section 12 above to obtain the distance from the given locality to Mecca. A single application of the law of sines then suffices to produce the required azimuth. The method stems ultimately from the Abbasid astronomer Ḥabash al-Ḥāsib ([24], p.42).

f.174r:9 (AS 131v:5) **Chapter 2. Proofs of the Above Operations**

Section 1. Declinations, and Ascensions of Transit

This has a single figure, employed for the proofs of both IV,1;1 and 2 above. Theorems are invoked which establish relations between the parts of a right spherical triangle, or pairs of them.

f.175v:19 (AS 133r:1) Section 2. Culminating Altitude and the Mean Sine

Here are proofs for IV,1;3 and 4 above. Now the work is done with projections on the meridian plane shown in a new figure. The surface of the sphere is for the time being abandoned.

f.176v:11 (AS 133v:5) Section 3. Azimuth from Altitude

This proves IV,1;7 above. Here also operations are carried out upon plane configurations inside the sphere.

f.177v:18 (AS 134v:11) Section 4. Altitude from Azimuth

This is the inverse of the problem just solved, hence it is a proof of the rules given in IV,1;8 above. Now the author reverts to proper spherical trigonometry, operating entirely on the surface of the sphere.

f.178v:8 (AS 135r:14) Section 5. Determination of Ascensions

This is a proof of the rules given in IV,1;9 . It is carried through by applying the Rule of Four to two pairs of spherical right triangles. The theorem is used only for first quadrant arcs, so it is necessary to add arrangements for instances when the argument is in one of the other three quadrants.

f.179r:19 Section 6. Determination of Inverse Ascensions

(The scribe copying onto AS seems to have left out this section when he came to the end of Section 5, and went right on with Section 7. Someone checking the manuscript noted the omission, and copied Section 6 (with its figure) around the margins of three sides of f.135v.)

Again this is the inverse of the problem just solved. So a proof is needed for the procedure of Section 10 above. To justify the right ascension inversion, the Rule of Four is invoked with a pair of spherical triangles. The result is used to set up a second relation, also validated by the Rule of Four. The second relation yields the desired inverse oblique ascension.

f.179v:23 (AS 135v:19) Section 7. Determination of the Distance Between Two Stars

This consists of proofs for the rules given in IV,1;12 above. They involve repeated applications of the Rule of Four. Unique to this problem is the large number of non-trivial special cases. As a result, the author found it necessary to present ten figures in this single section.

f.181r:13 (AS 136v:21) Section 8. Azimuth of the Qibla

The material of IV,1;15 is proved here, with the aid of the ubiquitous Rule of Four.

Notice that Chapter 2 omits all mention of Sections 5, 6, 11, 13, and 14 of Chapter 1, Kāshī claiming that the material they contain is self-evident. This includes Section 11, where much of what he says is wrong.

f.183r:2 (AS 137v:13) TREATISE V. DETERMINATION OF THE ASCENDENT FROM VARIOUS COMBINATIONS OF OBSERVED QUANTITIES

(The contents of this treatise have been described in [27]. To cast the birth horoscope of a person, the astrologer must know the *ascendent (ṭāli*ᶜ*)* at the instant of the birth. The ascendent is the point on the ecliptic crossing the eastern horizon at the time and place of the birth.)

Introduction: A Technical Vocabulary

This is not so much a list of definitions as a set of descriptions of observational instruments, particularly the rotating triquetum, but also including the hand quadrant, the mural quadrant (*libna*), the cup clepsydra, and the plumb line.

However, there are also definitions: that of the *arrangement sine* (*h*Sin, *jayb-i tartīb-i dāyir*, [24], p.45), hour angle, total and partial parallax.

f.183v:13 (AS 138r:1) **Chapter 1. Rules for Carrying Out Operations**

Section 1. Various Techniques for Determining the Ascendent

The operations sketched are all observational, employing the instruments previously described, also any handy vertical wall. The subsequent mathematical reductions are detailed in the following sections.

f.184v:9 (AS 138v:11) Section 2. Determination of the Ascendent from the Altitude (Proofs in V,2;1 and 2 below.)

Four equivalent rules are given for calculating the hour angle in terms of the star's altitude and its culminating altitude on the day of the observation. From the hour angle and the star's ascension of transit the ascendent is obtained by the use of normed right ascensions.

f.185r (AS 139r) **Table of Maximum Parallax in Altitude for the Sun, Moon, and Venus** (Cf. V,2;2.)

For the sun the argument is each ten degrees of mean longitude from the apogee. For Venus and the moon, the arguments are distances from the earth. All entries are to seconds.

f.185v:11 (AS 139v:1) Section 3. Determination of the Ascendent of a Star from Its Azimuth (Proofs are in V,2;3.)

From the azimuth calculate the altitude as explained in IV,1.8 Thence proceed as in the preceding section.

f.186r:1 (AS 139v:4) Section 4. Determining the Ascendent by Marking the Shadow Cast by a Gnomon (Proofs in V,2;4.)

Four situations are considered with respect to the gnomon and the plane, level or otherwise, on which its shadow is cast. For each situation a technique is described for obtaining the shadow length. From the latter the altitude is determined trigonometrically. With the altitude, proceed as in Section 2 above.

f.186v:8 (AS 140r:7) Section 5. Determining the Ascendent from a Mark Indicating the Direction of a Shadow

The shadow length not being immediately available, various expedients are described, either to construct the horizontal shadow and gnomon lengths, or to obtain the azimuth. Then use Section 3 or 4 to find the ascendent.

f.187r:6 (AS 140r:23) Section 6. Determining the Ascendent from the Shadow of a Peg Driven into a Wall

Again, for various situations of the peg, the procedure is to construct physically two legs of a right triangle. On one a plumb line, say, the gnomon length is measured. The other may be simulated by a mason's level to give the shadow length. Eventually apply Section 2 above.

f.187r:17 (AS 140v:11) Section 7. Extracting the Ascendent by Marking the Shadow of the Top of a Wall, the Top Being Horizontal

This method is ludicrously impractical. Once the measurements have been made, they may be reduced by an algebraic rule which the author claims he has produced, but it would spoil the fun for the user if he were told. The solution is left to him. The challenge was met by M.-Th. Debarnot and published in [13].

f.189r:18 (AS 142v:10) Section 8. Determining the Ascendent from the Shadow of an Inaccessible Object

This seems to be solved with a practical and clever construction involving a string and a ruler. It is fully described (in [27], pp.131,132), with a figure, there being none in the zīj.

f.190r:3 (AS 143r:14) Section 9. Determination of the Ascendent After the Lapse of Four Julian Years

Now the astrologer is invited to apply the fact that four Julian (or Yazdigird) years are very close to four solar years, so that four such years after the nativity the situation replicates itself. If it is cloudy, never mind – wait another four years and try again.

f.190r:11 (AS 143r:20) Section 10. Determination of the Ascendent When Two Stars Have the Same Azimuth (Proofs in V,2;5)

An observer can determine with no other instrument than a plumb line when the given condition is satisfied. However, to determine the ascendent from this, involves the successive trigonometric computation of five auxiliary quantities.

f.191v:1 (AS 144r:20) Section 11. Determination of the Ascendent by the Use of Clepsydras

If an event occurs when the sky is clouded, the observer may measure with a water clock the time elapsed until the weather clears, then observe and calculate the ascendent. The technique described, counting successive cupfuls of water, seems crude.

f.191v:14 (AS 144v:6) Section 12. Determination of the Ascendent When a Star Is in the Meridian or on the Horizon

If the luminary is the sun, an application of normed right ascensions suffices. For other objects, reduction becomes more elaborate.

f.192r:23 (AS 144v:26) Section 13. Determination of the Ascendent for Horizons of the Afternoon, Evening, and Morning Prayers

The Muslim prayer times are defined astronomically. Perhaps the idea is that if someone remembered that a birth coincided with the muezzin's call, this would enable calculation of the ascendent.

f.192v:10 (AS 145r:10) Section 14. On the Determination of the Ascendent from the Namūdārs (Demonstration in V,2;6.)

If the time of the nativity is known only approximately, there were techniques called *namūdārs* (Latin *animodar*) which purported to enable a precise determination.

f.193r:1 (AS 145r:23) The Namūdār of Hermes the Sage

The procedure is based on a single postulate: the lunar longitude at the instant of birth is the ascendent at the instant of conception, and conversely the lunar longitude at conception is the ascendent of the nativity. From approximate initial information a convergent iterative algorism was worked out leading to a unique solution satisfying the postulate. This is of considerable mathematical interest.

f.193v:11 (AS 145v:25) The Namūdār of Zoroaster the Sage

The procedure is ill defined, depending upon astrological conditions at the approximate time of the nativity.

f.194v:1 (AS 146r:4) **Chapter 2. Geometric Proofs of the Above Operations**

Section 1. Determination of the Ascendent from the Altitude (Proofs for V,1;2.)

The rules proved here are worked out from plane configurations, probably of ultimately Indian origin.

f.195r:16 (AS 146v:15) Section 2. Determination of the True Altitude from the Observed Apparent Altitude

Here the parallax theory is presented upon which the table of f.185r is based.

f.195v:20 (AS 147r:16) Section 3. Determination of the Hour Angle from the Azimuth (Proofs of V,1;3.)

For the demonstrations, Kāshī returns to the surface of the sphere with the tangent case of the Rule of Four.

f.196r:15 (AS 147v:2) Section 4. Determination of the Ascendent from a Gnomon Shadow Cast on a Horizontal Plane (Proofs of V,1;4.)

The tangent case of the Rule of Four crops up again here.

f.196v:23 (AS 148r:5) Section 5. Determination of the Ascendent from Two Stars Having the Same Azimuth (Proofs for V,1;10.)

The demonstration runs through a succession of six trigonometric equations involving applications of the law of sines and the second case of the substitution theorem (*shikl i-mughnī*).

f.197v:13 (AS 148v:13) Section 6. Difficulties With the Namūdār of Hermes

A detailed description of the algorism.

f.198v (AS 149v:1) **TREATISE VI. ON THE REMAINING ASTROLOGICAL OPERATIONS**

Introduction: Technical Glossary

The first definition of the list is that of *incident horizon (ufuq-i hādith)*, a concept much applied in astrology, particularly in the Persian zījes. For a given star, it is the great circle through the star and the north and south points on the local horizon. Most of the remaining definitions involve operations with the incident horizon. Of these, two introduce the important concepts of tasyīr (Greek *aphesis*) and fardār (Latin *firdaria*).

f.199r:6 (AS 149v:23) **Chapter 1. Rules for the Operations**

Section 1. Determining the Latitude of Incident Horizon

This, consistently with other latitudes, is defined as the distance of the incident horizon from the celestial north pole. The right ascension and declination of the given star being known, the determination involves a succession of verbal rules, applications of the Rule of Four and the sine theorem, the calculation of four auxiliary quantities, the star's horizon coordinates, and eventually the desired latitude of incident horizon. A partial proof, using a figure, is to be found in Chapter 2, at ff.203r:8 - 204r:17.

The section also gives an alternative method of calculating the latitude, as long and complicated as the first.

f.199v:23 (AS 150v:8) Section 2. Determining the Verified Ascension (*maṭālaᶜ muṣaḥḥa)*

The verified ascension is the intersection of the incident horizon with that half of the celestial equator which is nearer the given star. Two rules are given for calculating it, employing such quantities as the equation of daylight. Here also, as in the preceding section, a partial proof is given, at 204r:18-23.

f.200r:11 (AS 150v:23) Section 3. Determining the Projections of the Rays (*maṭāriḥ al-shuʿāʿāt*)

If two planets on or near the ecliptic are separated by certain fixed distances they are said to be in aspect: sextile (±60°), quartile (±90°), trine (±120°), and one planet may "project its rays" upon the other. This topic is treated exhaustively in Arabic by Bīrūnī in [5], pp. 1377-1393, abstracted for readers of English in [23]. Bīrūnī's methods are described in [16]. In the zīj Kāshī commences with a procedure he attributes to Ptolemy, as does Bīrūnī. The two are different, however, for Bīrūnī modifies the twelve numbers given above by subjecting the planetary longitudes to right and oblique ascensions and their inverses, whereas Kāshī applies verified ascensions, hence the incident horizon.

Kāshī then describes a second method which he attributes to the astrologers (*aḥkāmiyān*). This involves the ascension of the planet's transit, verified ascensions, and normed right ascensions (81v:4).

f.200v:1 (AS 151r:9) Section 4. On Nativity Tasyīrs

The tasyīr of any person is an arc of the ecliptic believed to contain points related to important events in his life, in particular the length of life (see [2], p.323). In the determination of the endpoints of the tasyīr described in the section, the concept of incident horizon plays a fundamental role.

f.201v:1 (AS 151v:25) Section 5. On Nativity Intihā's

There are many varieties of intihā', but each is generated by a point moving along the ecliptic at constant speed. The position at which the point arrives at the end of some unit of time, say a year, is the intihā'. A year-intihā', for instance, travels one zodiacal sign per year. There are also month and day intihā's. The initial position of the generating point is determined by some event. Thus, for a person consulting an astrologer, an important year-intihā commences at the ecliptic point occupied by the sun at the instant of his birth. The position of the intihā' at any subsequent time was believed to indicate conditions at that time. This was done by, say, examining the attributes of the planet which rules the zodiacal sign into which the intihā' had just entered.

f.201v:10 (AS 152r:10) Section 6. On Nativity Fardārs and the Years Allotted to the Planets

This section describes a procedure attributed to Māshā'allāh, the celebrated eighth century Jewish astrologer. According to this scheme, the life of a native is divided into periods of years, each ruled by one of a set of nine celestial objects. This attributes to the sun ten years, to Venus eight years, to Mercury thirteen, the moon nine, Saturn eleven, Jupiter twelve, and Mars seven. To the ascending and descending lunar nodes three and two years respectively, so that the life span totals seventy-five years.

The set can also be arranged as a cycle in the order of decreasing size of successive orbits, omitting the nodes. So Saturn will be followed by Jupiter, then Mars, and so on, returning eventually to Saturn. The complicated manner by which the scheme is applied to individual natives is described below in the explanation of the table on f.208v (see [34], p. 62). The table is from the Īlkhānī Zīj. The author remarks that some astrologers prefer a different scheme, in which the intervals total a life span of ninety-four years. This is the sum of intervals given in the text, but Kāshī asserts that they add to ninety-eight. There is no table for the second scheme.

f.201v:23 (AS 152r:23) Section 7. On the Tasyīrs, Intihā's, Fardārs, and Cycles Connected with the Horoscope of the Universe

Analogously to the life of a person, the entire span of the universe, taken to be a world-year of 360,000 years, was thought to be divisible into a complicated system of periods. According to Bīrūnī, in [5], this doctrine of world indicators was assembled by Abū Maᶜshar (Albumasar, c. 850), the most famous astrologer of the Middle Ages.

The middle of the span, said to coincide with the Flood, is in fact the Indian Kaliyuga era (17 February, -3101 A.D.). Three sets of indicators are assumed: world tasyīrs, intihā's, and fardārs. For each of these there are four subdivisions: the mighty (*aᶜẓam*, fem. *ᶜuẓmā*), the big (*akbar*, fem. *kubrā*), the middle (*awsaṭ*, fem. *wusṭā*), and the small (*aṣghar*, fem. *ṣughrā*). The subdivisions of the world tasyīr are called *qisma* (share or portion); the divisions of the intihā's and fardārs retain these names. In addition to these, at f.202v:23, a cycle (*dawr*, pl. *adwār*) of 4,590 years is defined, made up by summing the mighty astrological gifts (*ᶜaṭāya*) of the planets thus: the sun 1461, Venus 1151, Mercury 480, moon 520, Saturn 265, and Mars 284 (cf. [34], p.28).

All the indicators pass through the first point of Aries at the beginning of the Kaliyuga, each at its own rate moving through the zodiacal signs. For each there is a table in the collection of tables which, as usual, follows Chapter 2 (ff. 203v - 213r). The rate at which each indicator advances is given below in the description of the tables showing the zodiacal positions of each as functions of time.

f.203r:8 (AS 153v:1) **Chapter 2. Proofs of the Above Operations**

There are no numbered sections. The chapter begins with a long (203r:9 - 204r:17) partial proof of the rules for calculating the latitude of incident horizon which are given in 199r:6 - 199v:22. Following this is a much shorter (204r:18-23) partial proof of the rules for determining the verified ascension given in 199v:23 - 200r:10. At this stage, Kāshī states that the determination of the projections of the rays is obvious, as are the rest of the operations of the treatise, and the chapter ends, on f.204v.

f.205v (AS 154v) **Table of Portions of the Days of the Year, for the Tasyīr of the Indicator of Nativities, One Solar Year (for) One Ascensional Degree**

The indicator here tabulated (defined at f. 200v:1-21) moves along the ecliptic at the rate of one degree per tropical year of length y = 365;14,32,30 days. (Other references in the zīj to a year imply one of this length unless a different year-length is specified.) The main table gives, to minutes and seconds of arc, the amount the indicator will have moved from the beginning of the year until the beginning of day d of that year. Hence the function is

$$f(d) = (d-1)/y, \quad \text{for } d = 1, 2, 3, \ldots, 366 \text{ days.}$$

rounded to seconds of arc.

A small auxiliary table gives the travel for hours. Here the function is

$$f(h) = (h-1)/24y, \quad \text{for } h = 1, 2, 3, \ldots, 24 \text{ hours.}$$

This format, which presents two tables of a single function on a single page, is used for many of the remaining tables.

Kāshī says this table has been taken from the Īlkhānī Zīj.

f.206r (AS 155r) **(Table of) Portions of Tasyīr Minutes in Days of the Year**

This gives successive "minutes", m, sixtieths of the year. So the function tabulated is

$$m(n) = ny/60, \quad \text{for } n = 1, 2, 3, \ldots, 60,$$

The minutes are expressed in months (of thirty days each), days, and hours.

(Table of) Portions of Tasyīr Seconds in Days of the Year

For this second table, the entries are "seconds", s, sixtieths of sixtieths, of the same y as above. So the entries are now

$$s(n) = ny/3600, \quad \text{for } n = 1, 2, 3, \ldots, 60,$$

expressed in days and hours.

f.206v (AS 155v) **Table of the Anniversary Tasyīr in Months and Days**

This indicator advances around the ecliptic, at constant speed, in such manner that it completes a revolution in the course of one year (ff.200v:22-201r:5).

For the main table the function is

$$f(d) = (d-1)(360^\circ/y), \quad \text{for } d = 1, 2, 3, \ldots, 366 \text{ days.}$$

The function is displayed in zodiacal signs, degrees, and minutes.

The small auxiliary table gives the travel in hours. So the function which produces it is $(h) = (h/24)(360^\circ/y)$, for $h = 1, 2, 3, \ldots, 24$ hours, displayed in minutes of arc.

f.207r (AS 156r) **Table of the Anniversary Centers Tasyīr, Obtained for Days, to Be Added to the Ascensions of the Transfer Ascendent**

In the course of a year this tasyīr is to travel, at constant speed, a revolution plus the excess of revolution, r = 87;15° starting from the vernal point (f.201r:10-22). To obtain the increment per day, the whole distance is divided by the year length. The upshot is that after 294 days have elapsed the tasyīr will complete a revolution, and be again at its starting point. At the end of the year it will be at r.

The function tabulated in the main table is

$$f(d) = (d-1)(r+360°)/y \ , \quad \text{for} \quad d = 1, 2, 3, \ldots, 366 \text{ days.}$$

The auxiliary table gives hourly increments. It is produced by the function

$$f(h) = (h/24)(r+360°)/y \ , \quad \text{for} \quad h = 1, 2, 3, \ldots, 24 \text{ hours.}$$

f.207v (AS 156v) **Table of the Annual Intihā'**

This table spreads out one sign, 30°, over the solar year. So the function tabulated in the main table is

$$f(d) = (d-1)(30°/y) \ , \quad \text{for} \quad d = 1, 2, 3, \ldots, 366 \text{ days.}$$

The auxiliary table gives the advance per hour. Hence the function is

$$f(h) = (h-1)/24(30°/y) \ , \quad \text{for} \quad h = 1, 2, 3, \ldots, 24 \text{ hours.}$$

f.208r (AS 157r) **Table of the Monthly Intihā'**

For this intihā' the motion in one year is thirteen signs, or 390° (f.201v:6-8). So its motion in days will be given by the function

$$f(d) = (d-1)(390°/y) \ , \quad \text{for} \quad d = 1, 2, 3, \ldots, 366 \text{ days.}$$

The auxiliary function gives the motion in hours, so its function is

$$f(h) = (h/24)(390°/y) \ , \quad \text{for} \quad h = 1, 2, 3, \ldots, 24 \text{ hours.}$$

f.208v (AS 157v) **Table of the Nativity Fardār**

The main table on this page has fourteen columns indicated by parallel lines below the title block. Column 1 gives the year numbers: 1, 2, 3, ... of a daytime birth. At year 31, however, the bottom of the page has been reached, and year 32 appears at the top of Column 6. The sequence of successive year numbers resumes, reaching 63 at the bottom of this column. For a third time the sequence resumes, now at Column 11, commencing with 64 and terminating with year 75. Seventy-five years is the maximum life span of a native in the cycle described in f.201v:10 above.

Since the years listed in Column 1 are for daytime births, the sun then being visible, the cycle is taken as commencing with the sun. In the cycle, the number of years allotted to the sun is ten. Hence the first ten years of the native's life have the sun as lord. This is indicated in Column 3 by the word "Sun" written vertically in a rectangle which takes up all of Column 3 opposite numbers 1 through 10 in Column 1.

In the order of decreasing orbits, Venus follows the sun, and is allotted eight years. So in Column 1, years 11 through 18 have Venus as their lord, and this is indicated by the word "Venus" written in a rectangle in Column 3 below the section for the sun. This continues with Mercury's thirteen years finishing Column 1, and its name in Column 2.

The years of lordship of the remaining planets, the moon, Saturn, and so on appear successively in Columns 6 and 11, their names in Columns 8 and 13, ending with the years assigned to the ascending node, and two years to the descending node. This completes the entries for diurnal births.

For nocturnal births the cycle commences with the moon. So the years 1 through 9 during which the moon is lord start at the top of Column 7, and proceed downward. The nocturnal years 10 through 20 for Saturn continue downward in Column 7, and so on through the cycle, using successively Columns 12 and 2. The planetary names already mentioned in Columns 3, 8, and 13 also serve for nocturnal births.

This completes the description of the main lordship divisions. Columns 5 and 10 are empty.

There remains the designation of the *associates (shurakā')*. These are the seven planets proper, omitting the nodes. The associates are supposed to exert minor influences on the life of the native during spans of time which are much shorter than the spans of integer years of the major lordships. They are in fact sevenths of the latter.

These are designated in Columns 4, 9, and 14, by a system which is complicated indeed. The description of the main lordships is sufficiently confusing, but it is simple compared with the associates. There is no word of explanation in the text of the zīj. The system has been inferred from the entries in the compartments of Columns 4, 9, and 14.

Each such entry gives either the name of a single planet written out, or it contains three Arabic letters. The middle letter is one of the *abjad* alphabetical numerals for 1, 2, 3, ..., 6. Each of these numerals appears once and only once in the span of years assigned to the lordship of each planet. The letter on either side of the numeral is the terminal (not the initial) letter of a planet. A particular planet may be designated in more than one compartment, but repetition never occurs more than once, and invariably in a lower compartment. The order is invariably that of descending orbit size.

For example, the span of lordship enjoyed by the sun is ten years, beginning with diurnal birth year 1. The span for each associate is a seventh of this, 10/7, a year and three sevenths. Of these subdivisions, the sun is associate for the first, Venus for the second, and so on, Mars being the associate for the seventh and last.

As a second example, the eight years of Venus' lordship provide spans of 8/7 years for each of the seven associates.

Years 71-75 of diurnal birth, ruled by the nodes, have no associates.

Columns 5 and 10 are empty.

A small auxiliary table occupies the space available under Columns 11-14 of the main table. These columns have only twelve entries, and do not reach the bottom of the of the page.

This table gives the lengths of 1, 2, 3, ..., 7 sevenths of a standard year, *y*, expressed in months of thirty days, days, and hours. The hours are carried to three fractional sexagesimal places.

The function of the table is to facilitate the computation of the fractions of years in the spans of the associates.

f.209r (AS 158r) **Table of Cycles of the Nativity Years**

This page, in both copies of the manuscript, has the title block and border only. The interior contains no entries.

ff.209v-210v (AS 158v-159v) **Table of the Tasyīrs, Intihā's, and the Mighty Fardār Between Three and Four Hundred Malikī**

The rates of advance for the mighty world tasyīr, the big, the little, and the small, are

$$1^{\circ}/1000^{y},\ 1^{\circ}/100^{y},\ 1^{\circ}/10^{y},\ \text{and}\ 1^{\circ}/1^{y}$$

respectively (f.202r:6).

As for the intihā's, for the same four categories, their rates are

$$1^{s}/1000^{y},\ 1^{s}/100^{y},\ 1^{s}/10^{y},\ \text{and}\ 1^{s}/1^{y}$$

respectively, where *s* stands for *zodiacal sign* (f. 202r:9-12).

The mighty fardār advances at the rate of $1^{s}/360^{\circ}$ (f. 202r:14).

All nine of these indicators set out from the first point of Aries at the epoch of the Flood (Kaliyuga).

The three pages of this table show the positions they have reached by the beginnings of Malikī years 301 (1379 A.D.), 302, 303, ..., 402. for each of these dates the entries give the positions of the

World tasyīrs: The mighty, in zodiacal signs and degrees to three fractional sexagesimal places.

The big, in signs and degrees to two sexagesimal places.

The middle, to one place.

The small, to integer degrees.

World intihā's: The mighty, in signs and degrees to seconds of arc.

The big, to minutes.
The middle, to signs and degrees.
The small, to integer signs.

The mighty fardār: in signs and degrees, to minutes.

f.211r (AS 160r) **Table of Motion of the Tasyīrs, Intihā's, and the Mighty Fardār in Hundreds, Thousands, and Tens of Thousands of Years**

As the title indicates, the arguments are
y = 100, 200, 300, ..., 900, 1000, 2000, 3000, ..., 9000, 10,000, 20,000, 30,000, ..., 90,000, 100,000 years.

The entries give the motions of the four types of tasyīrs, the four intihā's, and the mighty fardār, in signs, degrees, and minutes.

f.211v (AS 160v) **Table of the Mighty Fardār**

The structure of this table is simple indeed. For an argument n = 1, 2, 3, ..., 84, the entries are values of

$$f(n) = 261 + 360(n-1) \text{ years.}$$

To explain this function, however, involves considerable background, including the definition of the mighty fardārs. This states that 360 years are required for each pair consisting of a zodiacal sign and a planet. The fardār commences, like all the indicators, at the epoch of the Flood (Kaliyuga) at the first point of Aries. At that time it is associated with the planet Saturn.

At the expiration of the first 360 years, the indicator will have moved to the first point of Taurus, and the second planet, Jupiter takes over. At the end of seven 360 year intervals, all the planets will have appeared, and Saturn returns. But now the other element of the pair will be the eighth sign, Scorpio. At the end of twelve intervals, the indicator will be back at Aries, but the planet will be Mercury.

Moreover, since twelve and seven are relatively prime, $12 \times 7 = 84$ intervals elapse before the combination (Aries, Saturn) recurs. Hence the highest value of the argument is 84. The numerical entries are in the decimal place value system, the numeral forms being those presently current in Iran and the Arab countries.

Behind each numerical entry is a pair of columns, the first indicating the successive signs by the repeated abjad numerals 0, 1, 2, 3, ..., 11. The second column designates planets by the terminal letters of their names.

Note that this fardār has the enormous span of $12 \times 7 \times 360 = 30{,}240$ years. The last entry in the table is 30,141.

The title of the table directs that, given a Malikī incomplete (*nāqis*, current) year, the user should add 3,000 to it and enter the table. This number will fall between a pair of adjacent entries. Within the entire 360 year span determined by the two entries there is a single combination of sign and planet given in the upper of the two rows so designated. This is valid for the whole interval.

The difference in the epochs of the Flood and of the Malikī calendar (15 March, 1070 A.D.), addition of 3000, and the initial tabular entry of 261 insure that the result will be correct.

f.212r (AS 161r) **Table of the Motion of the Mighty Fardār in Years**

The argument of this table is years, y, which run through the set

$$y = 1, 2, 3, \ldots, 59, 60, 120, 180, 240, 300, 360,$$

represented in abjad non-place value numerals.

The function tabulated is $f(y) = 0;5^{\circ}y = y/12^{\circ}$.

The entries are in abjad sexagesimals.

The maximum entry in the table is $f(360) = 30^{\circ}$, a zodiacal sign. A property of the mighty fardār (f. 202r:14) is that it traverses a sign in 360 years. Hence it is clear that the table gives the motion of the fardār in degrees per year.

If the argument is a number greater than 60 and less than 360, subtract successive sixties from it to find a remainder less than 60. Find the entry in the table having the remainder as argument. The number from which the remainder has been subtracted will be a member of the set 60, 120, 180, 240, 300. Look up its tabular value, and add it to the value previously found.

Table of Positions of the Big Fardār

The argument of this table is the set 1, 2, 3, ..., 78 years, represented in non-place value abjad numerals.

The entries in the table are numbers, each displayed as a sequence of signs, degrees, and minutes, all in abjad form. If plotted as a graph, it would appear as a sequence of straight line segments of continually decreasing lengths, but continually increasing slopes. This broken line starts at the origin, and ends with the 78 year period at a height of eleven signs ($11 \times 30^{\circ} = 330^{\circ}$).

To explain the complicated, not to say perverse, system underlying the computations, is necessary to state a property of the number 78. It is the sum of the first twelve positive integers. That is

$$1+2+3+ \ldots +12 = 78.$$

Commencing from zero, the first twelve entries have a constant increment of $1^{s}/12 = 30^{\circ}/12 = 2;30^{\circ}$. Hence $f(2) = 2;30^{\circ}$, $f(3) = 5;0^{\circ}$, and so on until $f(13) = 1^{s}\ 0;0^{\circ}$. Now, however, the sign to be added is split eleven ways, $30^{\circ}/11 = 2;44^{\circ}$ to be added successively to the next eleven entries. So $f(14) = 1^{s}\ 2;44^{\circ}$, and $f(24) = 2^{s}\ 0;0^{\circ}$.

Next the sign is split ten ways, giving an increment of $30°/10 = 3°$ to be added to the next ten arguments, and so on. The table ends at $f(78) = 11^s\,0;0°$.

Since this indicator commences at the Flood epoch, it remains to carry out a norming operation so that the table will produce correct results when used for Malikī years. To this end, it is prescribed that, given a Malikī year, the user first add to it two years, then subtract from the sum successive 78's until the remainder is less than 78. Enter the table with this as argument.

ff.212v,213r (AS 161v,162r) **Table of the Middle and Small Fardārs and the Associates** .

These two pages contain, in fact, two tables. The first, and by all odds the largest has, running down the side of both pages, a column of the argument: 1, 2, 3, ..., 75 in non-place value abjad numerals. The table proper has nine columns, each with 75 compartments, thus a total of 675. These are filled with entries in decimal place value numerals of the kind presently current in Iran and the Arab world.

The column headings are, in the order of the exaltations: 1.The sun, 2 The moon, 3.The ascending node, 4.Jupiter, 5.Mercury, 6.Saturn, 7.The descending node, 8.Mars, and 9.Venus. So any entry in Column 4, for instance, indicates a year during which Jupiter was lord.

The entries of Column 1 are 96, 97, 98, ..., 170 (=96+75). On this basis, it would be expected that the entry at the top of Column 2 would be 171. This is not the case, for entry 171 has been displaced 10 compartments down, and appears opposite argument 11. At this point the sequence of entries, each increasing by one, resumes, and continues to the bottom of the column. Here the last entry is 235. Again, it might be expected that 236 would appear at the top of Column 3. It is indeed at the head of a column, but it is Column 2, not 3. So Column 2 commences with 236, to which successive ones are added below, until all the empty compartments are filled. This occurs opposite argument 10, with entry 245. This curious practice, of an incomplete column completing itself at its own top, is characteristic of all the remaining columns in the table.

Moving on to Column 3, the highest entry thus far made, 245, must be followed by 246. It is inserted in Column 3, but not at the top. This time the displacement is opposite argument 20, an additional 9 compartments to the previous 10 from the top. From there the entries increase by ones to the bottom of the column, which has entry 301. Entries 302 through 320 are inserted from the top, thus completing the column.

In Column 4, entry 321 is placed opposite argument 23, an additional 3 compartments to the previous 19 from the top. From there the entries again increase by ones to the bottom of the column where the entry is 373. So Column 4 commences with entry 374, to which successive ones are added until the empty compartments are filled, and the column is complete.

In this manner the entire table is completed. The successive increases in the displacement of the entries are: 10, 9, 3, 12, 13, 11, 2, and 8. These are the divisions of the life of a native utilized in the nativity fardār (see f.201v:10).

Note that the maximum entry in the table is 770 (=95+675). Because of the displacements of the entries, this does not appear at the end of the table. It is in the last column, opposite argument 68.

To use the table, given a Malikī year greater than 770, the text prescribes that successive 675's be subtracted until the remainder is 770 or less. Find the remainder as an entry in the table. The planet named at the head of the column in which the entry is found is the lord of that year in the middle fardār.

The second table has to do with the small fardār. It takes up two additional columns at the opposite side of the page from the argument. The first of these is divided into nine long compartments. Each of these compartments in succession contains the name of the planet which appears in the first table at the head of each column. The length of each compartment corresponds to the number of years attributed to each planet in the nativity fardār. These spans of years add up to 75, which is the number of rows in the first table. So the lower boundary of the first compartment, that of the sun, is an extension of the line just below argument 10, and so on.

The second column of the table divides into sevenths each of the seven compartments of the first column which appertain to the seven planets proper. Each of these small compartments contains either one, or more often three, Arabic letters.

To use this table, given a Malikī year, find as described above the corresponding entry in the first table. This entry will be opposite a particular long compartment in Column 1 of the second table. The planet named in this compartment is the lord of the small fardār current during the given year. The same entry will be opposite a particular small compartment in Column 2 of the second table. One of the Arabic letters in the compartment designates the associate of the small fardār during the given year. The method by which this is arrived at is very complicated, and is sketched in the description of the table of the nativity fardār at f.208v.

At the bottom of f.213r is a note stating that the book was completed during the year 816 H. (1413/4 A.D.), and the copy in the year 905 H. (1499/1500 A.D.). There is no other colophon.

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الرياضيات الإسلامية والفلك الإسلامي

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أدوارد س. كندي

الزيج الخاقاني

لجمشيد بن غياث الدين الكاشي

محتواه وأهميته

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